Trigonometry, Common Ratios

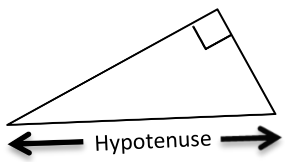
The word gives it away. *Trigonometry* is literally *the measurement of triangles*. Triangles are closed plane figures that have three angles and three straight-line sides.

Almost the first thing we learn about triangles is that their internal angles sum to two right angles. For a convincing demonstration of this, draw any triangle on a piece of paper. Cut it out then remove the three corners and place them side by side so that the angles abut each other. No matter what triangle is drawn, the three angles together create a straight angle.

This property of triangles has an application for polygons generally. For, in a polygon with n sides, the diagonals from one vertex to each of the others can be drawn to form $n-1$ triangles. So, the internal angles of an n-sided polygon must add to $(n-1) \times 180$ degrees.

Many commonly encountered triangles contain a right angle. When that happens, it can be useful to observe that the two smaller angles are *complementary*, meaning that together they too form a right angle.

The longest side of a right-angled triangle is called the *hypotenuse*, a Greek word meaning to *stretch underneath* the two other legs, as shown in this diagram.

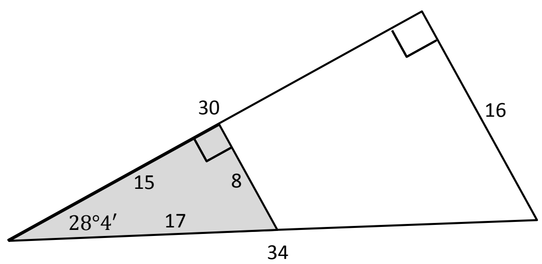


The hypotenuse is the side opposite the right angle. Its length is always less than the combined length of the other two sides, but remarkably, the sum of the squares of each side is exactly equal to the square of the hypotenuse.

This is *Pythagoras' Theorem*. So, a right-angled triangle with smaller sides 8 and 15 has a hypotenuse of length $\sqrt{8^2+15^2}=17$ units. There are many proofs of this beautiful relationship; so many that it would take a large book to contain them all.

The Greeks also noticed that a triangle with, for example, sides 8, 15 and 17 had the same three angles as a triangle of sides 16, 30 and 34. In fact doubling or tripling or indeed multiplying the side lengths by any positive number at all, leaves all the angles unchanged.

In the two triangles in the diagram below, the smallest angle is about $28°4'$ and it lies opposite the smallest side for each triangle.



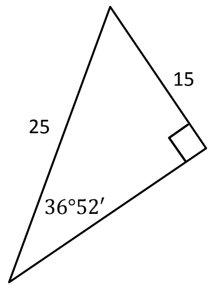
These triangles happen to be right-angled triangles, as can be confirmed with Pythagoras' theorem.

Knowing the angle $28°4'$ in these right-angled triangles does not help us to determine the size of any of the sides of either triangle. However, it *does* allow us to determine the *ratios* that exist between pairs of sides within each triangle. For example, if we know that in a particular right-angled triangle the ratio of the smallest side length to the next smallest side length is $\frac{8}{15}$ (or $\frac{16}{30} = \frac{8}{15}$) then we know that one of the angles must be $28°4'$. Conversely, if there is an angle of this size in a right-angled triangle, there must be sides in the ratio $\frac{8}{15}$.

Sine and cosine

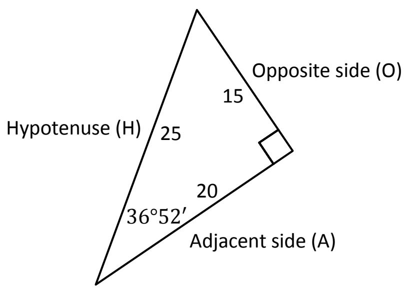
Eventually, names were given to the ratios that can be formed by dividing one side by another. One of these is *sine*. A sine ratio has been worked out for angles increasing in small steps between $0°$ and $90°$. It is a ratio between a right-angled triangle's shorter side (it does not matter which) and its hypotenuse. For example, if the sine ratio was $\frac{3}{5}$, then a sine table (or a scientific calculator) informs us that there is an angle very close to $36°52'$ in the triangle.

The *complementary* angle must be $53°08'$ and so all three angles can be determined from just one ratio. Note that we cannot as yet determine any lengths, but we can at least construct examples from the set of all possible triangles with the required ratio. One of them is shown here.



By Pythagoras' Theorem, the other side length in this example is $\sqrt{25^2 - 15^2} = 20$, and this might lead to a point of confusion: What if the sine ratio was constructed using the *other* shorter side? What if the sine ratio was $\frac{20}{25} = \frac{4}{5}$ instead? The table for sines shows that $\frac{4}{5}$ is associated with the angle $53°08'$, and this larger angle's *complement* is $36°52'$. The three angles are just oriented differently.

To avoid any confusion with shorter sides, a second complementary ratio was developed called *cosine*, and the shorter sides came to be labelled according to their position relative to the given angle. The shorter side *opposite* the angle in question became known as the *opposite side* and the shorter side adjacent to the angle in question was called the *adjacent side*.



Thus, the sine of the angle $36°52'$ in the diagram is defined as the ratio of the opposite side to the hypotenuse or simply $\frac{O}{H}$, and the cosine of the same angle is defined as the ratio of the adjacent side to the hypotenuse or $\frac{A}{H}$. In modern notation, using shortened forms of the words sine and cosine, we say that the $\sin 36°52' \approx \frac{3}{5}$ and the $\cos 36°52' \approx \frac{4}{5}$.

It is useful to remember however that sine and cosine are complementary ratios in that the sine of an angle is the cosine of the complementary angle – in symbols, $\sin\theta = \cos(90° - \theta)$.

Using trigonometric ratios

Consider the right-angled triangle whose hypotenuse has length of 16 cm and which contains an angle of $40°$. Can the triangle's perimeter be found? The answer is *yes*.

Since $\sin 40° = 0.642788$, according to a book of 6 figure tables, we know that $0.642788 = \frac{\text{Opposite}}{16}$ and therefore the opposite side is $16 \times 0.64278 \approx 10.285$ cm.

Then, Pythagoras' Theorem could be used to conclude that the other shorter side was $\sqrt{16^2 - 10.285^{\,2}} = 12.256$ cm.

Alternatively, by the cosine ratio we have $\cos 40° = \frac{\text{Adjacent}}{16}$ and thus, $\text{Adjacent} = 16 \times \cos 40°$, which a calculator simplifies to 12.257 cm. Thus the perimeter of this triangle is approximately 38.54 cm.

Tangent

The final part of the story is the third commonly used ratio called the *tangent of the angle*, or simply $\tan\theta$. It is simply the ratio $\frac{\text{Opposite}}{\text{Adjacent}} = \frac{O}{A}$. Noting that $\frac{O}{A} = \frac{O}{H} \div \frac{A}{H}$ we see that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and so the three ratios are interconnected.

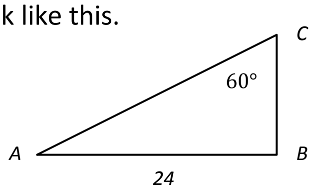
There are many problems for which the tangent ratio is useful. We can, for example, find heights of buildings or trees or widths of rivers without directly measuring them.

As an illustration of the power of this mathematical tool, suppose, when an observer is standing 18 metres from the base of a high-rise building, the angle of elevation measured to the top of the building is 78 degrees.

Then $\tan 78° = \frac{\text{height}}{18}$ and thus the height is $18 \times \tan 78°$ or 84.68 m. The reliability of this solution depends on the accuracy of the angle measurement, which becomes critical when using the tangent ratio with large angles. The distance from the building would also need to carefully measured, but the observer was able to find the height without climbing up the side of the building!

Questions

1. Pythagoras' theorem states that in any right angle triangle, the square on the hypotenuse is equal in area to the sum of the areas of the two squares that are built on the shorter sides. If the two shorter sides are equal, and each is of length s, by what factor is the length of the hypotenuse longer than the length of another side?
2. Because $\sin\theta = \frac{O}{H}$ and $\cos\theta = \frac{A}{H}$, write an expression for $[\sin\theta]^2 + [\cos\theta]^2$ and simplify it, using the fact that $H^2 = O^2 + A^2$ from Pythagoras' Theorem.
3. Generally speaking, we tend to label the vertices of a triangle with capital letters, like A, B and C and the sides opposite these angles as a, b and c. So we might consider a right angle triangle ΔABC with $\angle B$ 90°, $\angle C$ 60° and side $c = 24$ cm. It might look like this.



   Use the sine ratio to find the length of b the hypotenuse.
4. For the triangle in question 3, find the length of a by two methods. Firstly, use the tangent ratio, starting with $\tan 60° = \frac{24}{a}$ and solving. Secondly, using your answer for the length of the hypotenuse b found in question 3, use Pythagoras' Theorem to find a.
5. Two large rocks are on opposite banks of a wide river. An imaginary line between the rocks runs due north and the river runs due east. A mathematician walks along the river from the rock on the south side, and counts 20 paces before stopping. Using her experience, from that point, she estimates the angle between the two rocks subtended by her to be 60°. Estimate in paces the width of the river.
6. There is an interesting right-angled triangle that has the side lengths of $1, 2$ and $\sqrt{3}$. The two smallest angles are exactly 60° and 30°. Use the lengths to draw the triangle to scale ($\sqrt{3} \approx 1.73$), and then use a protractor to verify the size of the angles. Write down the six values for $\sin 30°$, $\cos 30°$, $\tan 30°$, $\sin 60°$, $\cos 60°$ and $\tan 30°$.
7. The diagonal of a rectangle is drawn and found to be 25 cm long. One of the angles that make up the two triangles formed by this diagonal is 30°. Find the area of the rectangle.

Answers

1. $\sqrt{2}s$
2. Simplifies to 1, since $H^2 = O^2 + A^2$ dividing by H^2 reveals the reason.
3. b =27.71
4. $a = 13.86$, and using Pythagoras, $a = \sqrt{27.71^2 - 24^2} = 13.85$, note rounding error on the length of b slightly changes answer.
5. $w = 20\tan 60 = 34.64$ paces
6. $\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}, \frac{\sqrt{3}}{2}, \frac{1}{2}$ and $\sqrt{3}$
7. 270.6 cm^2