A triangle with regions formed by Cevians.

A teacher, Tom-P, with an interest in preparing students for Mathematical Olympiad competitions, published the following problem.

> In a triangle with unit area but otherwise arbitrary dimensions, two sides are trisected and four cevians are inserted, as shown below.
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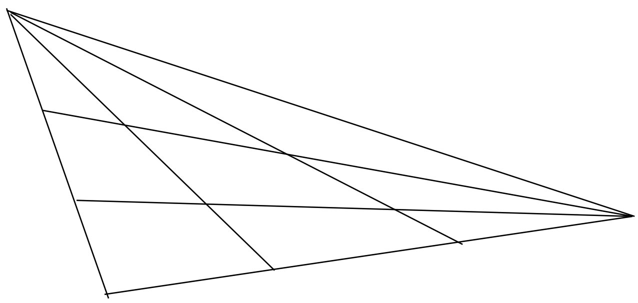
> 
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> What is the area of the quadrilateral formed in the centre of the triangle?

The term *cevian* comes from the theorem of Giovanni Ceva, discovered in 1678, about necessary and sufficient conditions for three such line segments - drawn from the vertices of a triangle and dividing each of the opposite sides in the same proportion - to be concurrent. Ceva's theorem is an idea somewhat related to the present problem but it does not appear to be immediately useful for its solution.

Implicit in the question is the assumption that the central region will occupy a fixed proportion in any triangle. That being the case, we might look for a triangle whose vertices and relevant intersections are in convenient locations to facilitate what could well be an untidy and difficult calculation. However, a convincing argument is needed to validate this assumption.

In another approach, we look to the theory of vector spaces. We recast elements from the initial diagram as vectors, as shown below. The idea will be to take two elements pointing in different directions as basis vectors so that every other element can be expressed as a linear combination of them. In this case, two sides of the triangle become the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

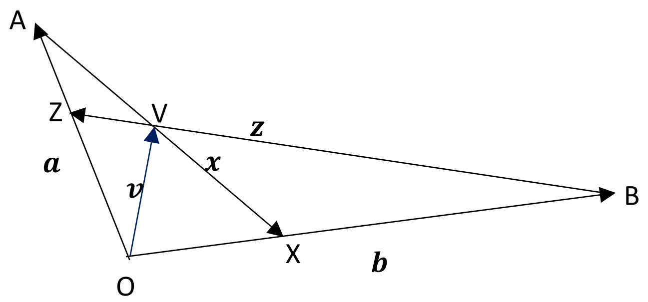
The area of a parallelogram can be expressed as a vector cross-product. That is, the area of the parallelogram with vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ forming its sides is given by $|\boldsymbol{u} \times \boldsymbol{v}|$.

In the given vector space, we will express $\boldsymbol{u}$ and $\boldsymbol{v}$ in terms of the basis vectors. Hence, we might write $\boldsymbol{u} = p\boldsymbol{a} + q\boldsymbol{b}$ and $\boldsymbol{b} = r\boldsymbol{a} + s\boldsymbol{b}$. Then, $|\boldsymbol{u} \times \boldsymbol{v}| = |(p\boldsymbol{a} + q\boldsymbol{b}) \times (r\boldsymbol{a} + s\boldsymbol{b})|$ and, by the algebraic properties of the vector cross-product, this simplifies to $|(ps - qr)(\boldsymbol{a} \times \boldsymbol{b})|$.

We derive some preliminary results that will facilitate the calculation.

Observe that, relative to the basis $(\boldsymbol{a}, \boldsymbol{b})$, the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ have coordinates (p, q) and (r, s) respectively, and that the expression $ps - qr$ is none other than the magnitude of the cross-product of the coordinate vectors. Thus, $|\boldsymbol{u} \times \boldsymbol{v}| = |(p, q) \times (r, s)||\boldsymbol{a} \times \boldsymbol{b}|$, which means the calculations to follow will depend on the coefficients independently of any specific triangle.

In the diagram below, point Z divides OA so that $\frac{\mathrm{OZ}}{\mathrm{OA}} = p$, and X divides OB so that $\frac{\mathrm{OX}}{\mathrm{OB}} = q$.



We have

$$\boldsymbol{x} = q\boldsymbol{b} - \boldsymbol{a}$$
$$\boldsymbol{z} = p\boldsymbol{a} - \boldsymbol{b}$$
$$\boldsymbol{v} = p\boldsymbol{a} - r\boldsymbol{z}$$
$$\boldsymbol{v} = q\boldsymbol{b} - s\boldsymbol{x}$$

Hence, after substituting for $\boldsymbol{z}$ and $\boldsymbol{x}$,

$$\boldsymbol{v} = (1-r)p\boldsymbol{a} + r\boldsymbol{b}$$

and

$$\boldsymbol{v} = (1-s)q\boldsymbol{b} + s\boldsymbol{a}.$$

The vector $\boldsymbol{v}$ has been expressed as a linear combination of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in two ways, but since these vectors are linearly independent, $\boldsymbol{v}$ can only be expressed as a linear combination of $\boldsymbol{a}$ and $\boldsymbol{b}$ in one way. Therefore, the coefficients must be equal. That is, $r = (1-s)q$ and $(1-r)p = s$. We deduce that

$$r = \frac{q(1-p)}{1-pq} \text{ and } s = \frac{p(1-q)}{1-pq}.$$

It follows that for each intersection we can find

$$\boldsymbol{v} = \frac{1}{1-pq}\left(p(1-q)\boldsymbol{a} + q(1-p)\boldsymbol{b}\right)$$

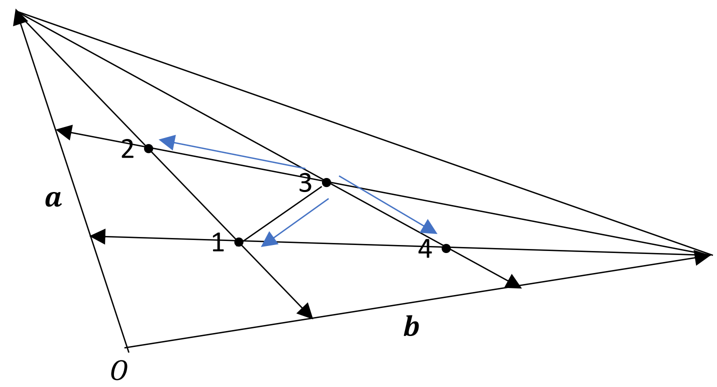
In the special case of intersecting cevians, the coefficients q and p are equal, and so are r and s. Provided $q \neq 1$, we then have $r = \frac{q}{1+q}$. Equivalently, $\boldsymbol{v} = \frac{q}{1+q}(\boldsymbol{a} + \boldsymbol{b})$.

[We remark in passing that this explains why the point of intersection of the medians of a triangle divides each median in the ratio $1:2$. In this case, $r = \frac{1}{3}$.]

These results, together with expressions for each of the elements in terms of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, make it possible to find the area of the central quadrilateral by dividing it into two triangles and using the vector cross-product.

The original diagram is reproduced below with the salient points of intersection numbered, and the triangles forming the central quadrilateral shown. The vectors that will be needed for the cross-products are also indicated.

In the present case, $q = \frac{1}{3}$ and $p = \frac{2}{3}$.



Relative to the basis vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we obtain from the expression for $\boldsymbol{v}$ given above, the following coordinates for the four points. The subscripts identifying the vectors $\boldsymbol{v_i}$ correspond to the numbered points.

$\boldsymbol{v}_1 = \left(\frac{1}{4}, \frac{1}{4}\right)$, $\boldsymbol{v}_2 = \left(\frac{4}{7}, \frac{1}{7}\right)$, $\boldsymbol{v}_3 = \left(\frac{2}{5}, \frac{2}{5}\right)$, $\boldsymbol{v}_4 = \left(\frac{1}{7}, \frac{4}{7}\right)$

Then, taking care to set up the cross-products so that the angles between the vectors are measured anticlockwise, we calculate

$$\frac{1}{2}[|(\boldsymbol{v}_2 - \boldsymbol{v}_3) \times (\boldsymbol{v}_1 - \boldsymbol{v}_3)| + |(\boldsymbol{v}_1 - \boldsymbol{v}_3) \times (\boldsymbol{v}_4 - \boldsymbol{v}_3)|]$$

That is, the area is

$$\frac{1}{2}[|\boldsymbol{v}_2 \times \boldsymbol{v}_1 - \boldsymbol{v}_2 \times \boldsymbol{v}_3 - \boldsymbol{v}_3 \times \boldsymbol{v}_1| + |\boldsymbol{v}_1 \times \boldsymbol{v}_4 - \boldsymbol{v}_1 \times \boldsymbol{v}_3 - \boldsymbol{v}_3 \times \boldsymbol{v}_4|]$$
$$= \frac{1}{2}\left[\left|\frac{3}{28} - \frac{6}{35}\right| + \left|\frac{3}{28} - \frac{6}{35}\right|\right] |\boldsymbol{a} \times \boldsymbol{b}|$$
$$= \frac{9}{140} |\boldsymbol{a} \times \boldsymbol{b}|$$

The area of the whole triangle is $\frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}|$. So, the proportion of the central quadrilateral to the area of the whole triangle is

$$\frac{9}{70}.$$

We divided the central quadrilateral into two triangles because the cross-product finds the area of a rectangle, not a general quadrilateral. But, we note that the two triangles have the same area anyway, and the simpler expression $|(\boldsymbol{v}_1 - \boldsymbol{v}_3) \times (\boldsymbol{v}_2 - \boldsymbol{v}_3)| = \frac{9}{140} \boldsymbol{a} \times \boldsymbol{b}$ leads to the same result.

There is a nice symmetry involved. The vector $\boldsymbol{v}_1 - \boldsymbol{v}_3$ is a scalar multiple of $\boldsymbol{v}_1$, which shows that the diagonal represented by $\boldsymbol{v}_1 - \boldsymbol{v}_3$ passes through the vertex O. The mid-point of the third side, given by $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$, expressed as a coordinate vector in the basis $(\boldsymbol{a}, \boldsymbol{b})$ is $\left(\frac{1}{2}, \frac{1}{2}\right)$ which is again a scalar multiple of $\boldsymbol{v}_1$, showing that the diagonal is a median of the triangle.