Cubic roots with a ruler

The cubic equation $ax^3 + bx^2 + cx + d = 0$ with real coefficients $(a > 0)$ can have up to three roots. If all of these roots are real their values are identified as x intercepts on the graph of the corresponding function $f\,(x) = ax^3 + bx^2 + cx + d$. If the cubic equation has exactly one real root, then it must have two complex conjugate roots, and these can also be identified using the graph of function.

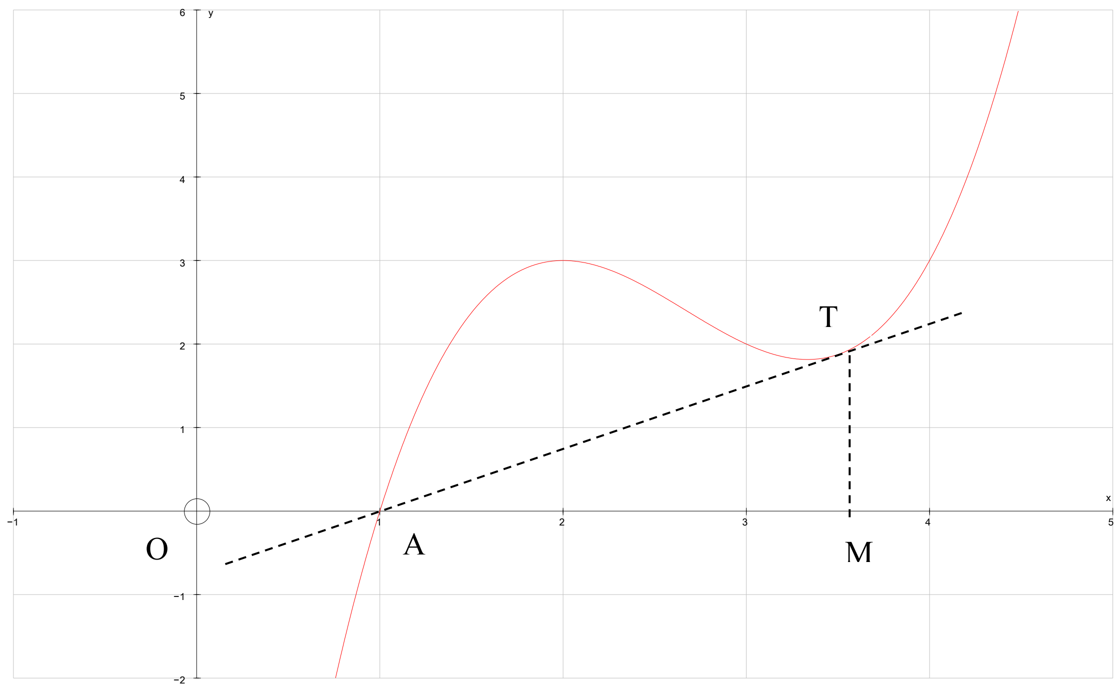
To do this we use the graph the function $g(x) = \frac{1}{a}f(x) = x^3 + rx^2 + sx + t$ having the same roots as $f\,(x)$.

The function $g(x)$ has the form $g(x) = (x - k)[x - (p + qi)][x - (p - qi)]$ and by expanding the second two factors we find $g(x) = (x - k)(x^2 - 2px + p^2 + q^2)$.

Suppose the line $y = \lambda(x - k)$ passing through the point $A(k, 0)$ at a gradient λ is also tangent to $g(x)$ at the point T where $x = t$. Then the value of t can be found by solving for x in the equation $\lambda(x - k) = (x - k)(x^2 - 2px + p^2 + q^2)$.

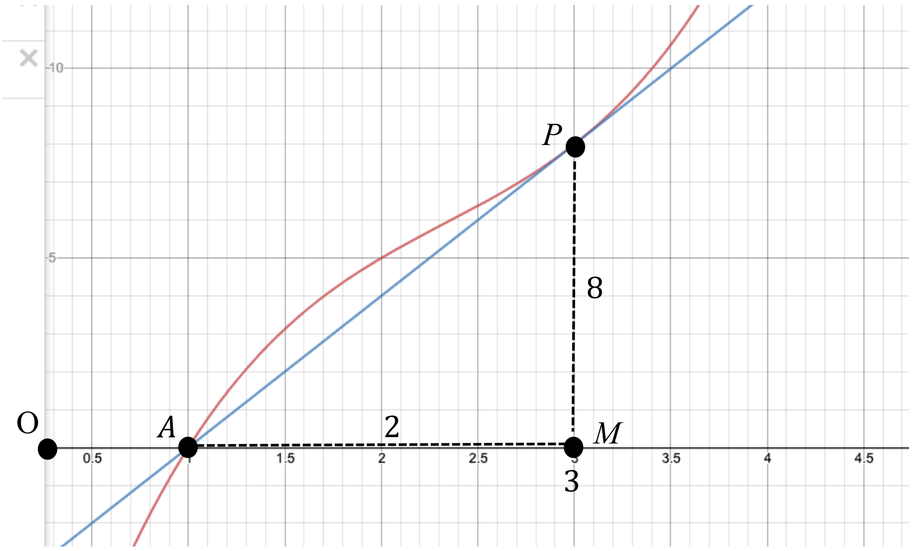
Thus we have by simplification that $x^2 - 2px + (p^2 + q^2 - \lambda) = 0$ and in the solution to the quadratic, because of the tangency condition, the discriminant Δ vanishes, so that $\Delta = 4p^2 - 4(p^2 + q^2 - \lambda) = 0$, and this shows $\lambda = q^2$.

This means, theoretically at least, that q can be found as $\sqrt{\lambda}$. The gradient λ of the line can be approximately determined by direct measurement as the ratio $\frac{TM}{AM}$ as shown in the illustration.



With $\Delta = 0$, the solution to the quadratic equation $x^2 - 2px + (p^2 + q^2 - \lambda) = 0$ is given by $x = \frac{2p}{2} = p$ and this implies that $OM = p$ is the real part of the complex roots.

As an example, here is the sketch of $g(x) = x^3 - 7x^2 + 19x - 13$



The tangent line AP results in $PM = 8$ and $AM = 2$ so that the gradient of AP is given by $\lambda = 4$. This means $q = \sqrt{\lambda} = 2$, and since $OM = p = 3$, the complex conjugate roots are $3 \pm 2i$.

Thus $g(x) = (x - 1)[x - (3 + 2i)][x - (3 - 2i)] = (x - 1)(x^2 - 6x + 13)$.

Readers might be alert to the fact that the author cunningly set these numbers up by thinking in reverse. That is, starting with $\sqrt{\lambda} = 2$ and $\lambda = 4$, choose $AM = 2$ and thus $PM = 8$. Choose $A = (1,0)$ so that $M = (3,0)$ and $p = 3$. The two complex roots are $3 \pm 2i$ and a little algebra shows $g(x) = (x - 1)(x^2 - 6x + 13)$.

Perhaps more realistically the technique is useful for another purpose. By simply looking at an appropriately scaled graph of a cubic function a reasonable estimation of the complex roots can be made. By imagining (or actually drawing) a tangent line from the x intercept to the curve it is possible to develop an estimate of its gradient either directly as a ratio or else calculating $\lambda = \tan\theta$ from a measurement of the angle the tangent line makes with the x axis.

Also note that applying the technique on a function of the form $y = (x - u)(x - v)^2$ with real roots u and v would result in a tangent line running along the x axis through $A(u, 0)$ and $M(v, 0)$ so that $\sqrt{\lambda} = 0$ and $p = v$.