Gamow's puzzle – a tale in three proofs

George Gamow's book *One Two Three…Infinity – Facts and speculations of Science* published in 1947 by Viking Press and reprinted by Mentor Books for the New American Library in 1953, included a problem about a lost treasure.

While the authors in this blog have added a little flavour of their own the essential features of Gamow's original puzzle remain completely intact. Readers are urged to learn more about this fascinating puzzle by using the references provided at the beginning of each proof. Here is our version.

Written on an old parchment inside a bottle I found last year while at the beach on holidays were hints to the location of a buried treasure on an island with geographical coordinates of Lat. $0°51'13''S$, Long. $169°32'11''\ E$.

It described an area where two tall trees (an Oak tree and a Pine tree) and a set of old broken gallows were located. The paragraph containing the instructions read as follows:

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Go to the Gallows and walk directly to the Pine
tree, counting your paces as you do. Once there,
turn 90° to the left and walk the same number of
paces away from the tree. Put a stake in the ground
at that point. Go back to the Gallows and walk to
the Oak tree counting your paces as you go. Turn 90°
to the right this time and walk out from the tree
the same number of paces. Put a stake in the ground
at this point also. The treasure is buried exactly
half way between the stakes.
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My curiosity got the better of me and so about a year ago, I chartered a boat and went to the island. Indeed, not too far from *St Daniel Sinclair*, an old abandoned church, there stood the two trees. The small island had no other trees growing on it so I figured this must have been the spot. Alas, the Gallows were completely gone – not one hint of a piece of wood or rope to be found anywhere! It seemed the treasure would be lost forever. I had neither the means nor the time to dig the whole area up. Before leaving however, I tried my luck. I picked a single spot near the trees and pegged it. That spot would be my Gallows. I carried out the parchment instructions to the letter and to my utter disbelief found the buried treasure – one hundred gold coins! I said a quiet prayer that day in the church before leaving. Clearly some higher power was at work.

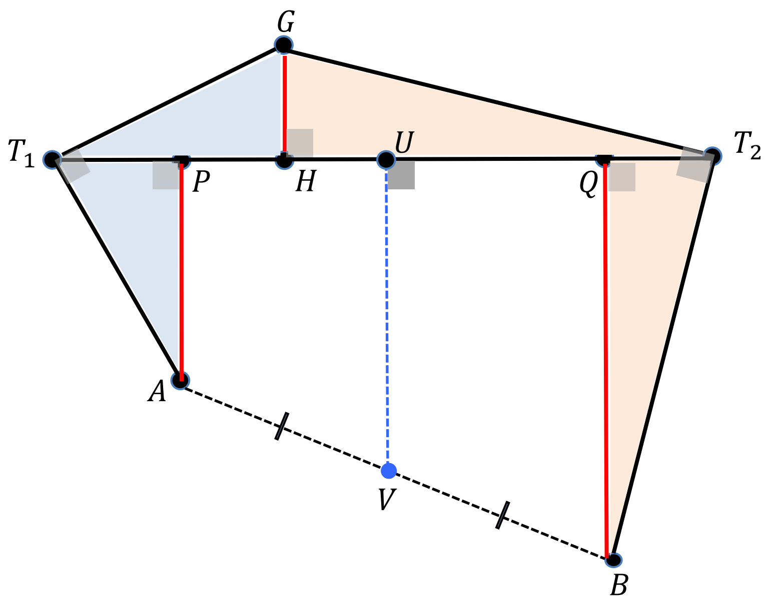
Recently though I managed to prove mathematically that divine intervention was probably not the reason I found the coins. Any point I might have chosen would have also led me directly to the treasure. While three different proofs of this are given below, I still wonder about why I had the idea in the first place to choose a random point.

Proof 1: Using plane geometry

This proof comes from Jim Wilson, University of Georgia – for more details check out the website http://jwilson.coe.uga.edu/emt725/Treasure/Treasure.html

The diagram below depicts two trees, T_1 and T_2, and an assumed point G where I imagined the location of the gallows. The stakes are shown at A and B, and the point V midway between A and B represents where the treasure was found.

Lots of other elements to the diagram have been drawn in. The line T_1T_2 shows internal points P, H, U and Q and the altitudes AP, GH, UV and BQ have been drawn. Four triangles have been highlighted as shown.



We will assume that G is above and between the fixed line T_1T_2. (The same proof would apply if G were below the line T_1T_2 but the diagram would be reversed)

Given: $GT_1 = T_1A$ and $GT_2 = T_2B$ with angles GT_1A and GT_2B right angles

Outline of proof:

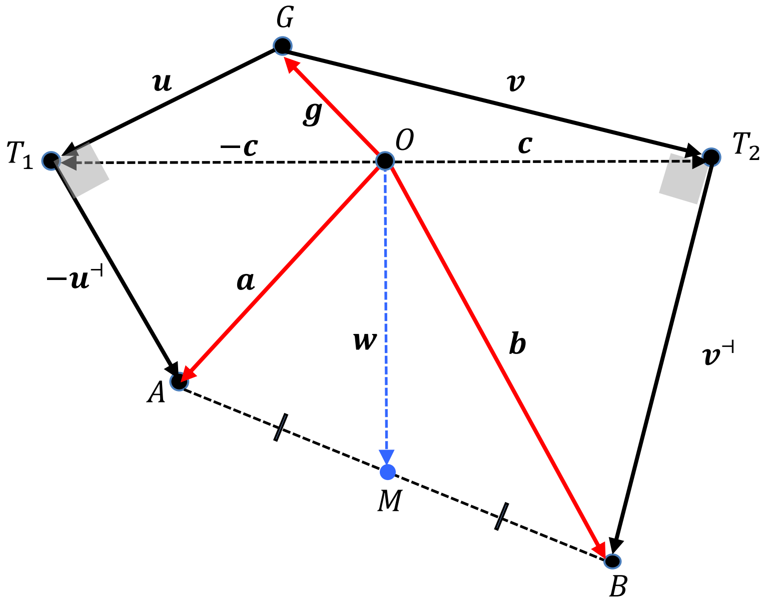
Triangle GHT_1 is congruent to triangle T_1PA (*Angle Side Angle*)
Triangle GHT_2 is congruent to triangle T_1QB (*Angle Side Angle*)
Hence the fixed distance $T_1T_2 = T_1H + HT_2 = PA + QB$
But the length of UV is $\frac{1}{2}(PA + QB) = \frac{1}{2}T_1T_2$

Hence the length and position of UV is fixed and does not depend on the position of G implying that the location of the gallows is irrelevant.

Proof 2: Using Vectors

Paul Turner derived this vector proof some years ago.

In the diagram small bolded letters are used as vector labels and the vector $\boldsymbol{v}^{\dashv}$ denotes a vector of length $|\boldsymbol{v}|$ rotated clockwise so that it is perpendicular to $\boldsymbol{v}$. The point O is the origin, placed midway between T_1 and T_2.



The vector $\boldsymbol{w} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) = \frac{1}{2}(-\boldsymbol{c} - \boldsymbol{u}^{\dashv} + \boldsymbol{c} + \boldsymbol{v}^{\dashv}) = \frac{1}{2}(\boldsymbol{v}^{\dashv} - \boldsymbol{u}^{\dashv})$

But $\boldsymbol{v}^{\dashv} = (\boldsymbol{c} - \boldsymbol{g})^{\dashv} = (\boldsymbol{c}^{\dashv} - \boldsymbol{g}^{\dashv})$
and $\boldsymbol{u}^{\dashv} = (-\boldsymbol{c} - \boldsymbol{g})^{\dashv} = -(\boldsymbol{c} + \boldsymbol{g})^{\dashv} = -(\boldsymbol{c}^{\dashv} + \boldsymbol{g}^{\dashv})$

So $\boldsymbol{w} = \frac{1}{2}[(\boldsymbol{c}^{\dashv} - \boldsymbol{g}^{\dashv}) + (\boldsymbol{c}^{\dashv} + \boldsymbol{g}^{\dashv})] = \boldsymbol{c}^{\dashv}$

In other words, the vector $\boldsymbol{w}$ is independent of $\boldsymbol{g}$, so the location of the point G is immaterial to the problem.

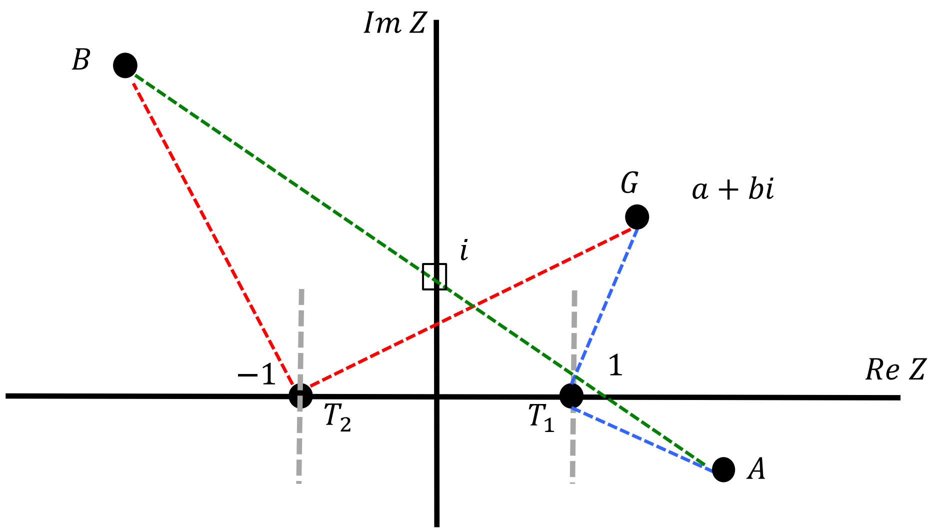
Note: it can easily be verified that, for vectors $\boldsymbol{p}$ *and* $\boldsymbol{q}$, $(\boldsymbol{p}^{\dashv} + \boldsymbol{q}^{\dashv}) = (\boldsymbol{p} + \boldsymbol{q})^{\dashv}$

Proof 3: Using Complex numbers

Sourced from Paul Nahin's book An Imaginary Tale – The story of $\sqrt{-1}$.

We can prove the solution invariance using complex numbers and the Argand diagram.

The Gallows are depicted as the complex number $a + bi$, and position the two trees T_1 and T_2 on the real axis at the unit distances $x = \pm 1$.



By a series of axes translations and rotations we can find the two staked positions B and A. We'll find B first, then A and then their midpoint.

Translating the axes horizontally to the left momentarily to T_2 (translated imaginary axis shown in light grey) will change G to $(a + 1) + bi$. Multiplying by i rotates the vector T_2G by $90°$ anticlockwise to T_2B so that, in respect of the translated axes, B becomes $i[(a + 1) + bi] = -b + (a + 1)i$. Finally, reverting to the original axes, B becomes $-b - 1 + (a + 1)i$.

Repeating the procedure with A we shift the axes to the right so that G becomes $(a - 1) + bi$. Rotating clockwise this time using – i as the operator changes A to $-i[(a - 1) + bi] = b - (a - 1)i$ in the translated axes, and hence by reverting back to the original axes we have A as $(b + 1) - (a - 1)i$.

The midpoint of BA becomes

$$\frac{-b - 1 + (a + 1)i + (b + 1) - (a - 1)i}{2} = i$$

The midpoint is entirely independent of the values of a and b, and once again, the position of the Gallows is irrelevant.