**Lill’s method and diagrams**

**How to draw them & why they work**

**A classroom resource**

**By Paul Turner and Ed Staples**

[**www.mathematicalwhetstones.com**](http://www.mathematicalwhetstones.com)

**Edourd Lill**

Edourd Lill (1830 – 1900) was an Austrian engineer joining the military engineering corps of the Austrian Empire and reaching the rank of captain in that capacity. Lill is best remembered for his mathematical work, in particular for the modeling of 19th Century traffic problems, but he also developed a striking mathematical technique for locating the roots of polynomial equations.

**Horner’s scheme**

The method, described below, has a correspondence with Horner’s representation of a polynomial. William Horner (1786-1837) was an English mathematician and headmaster of Kingswood School, Bristol, and he had devised a way of minimizing the working required to evaluate a polynomial at any given value of the independent variable. And in those days, such discoveries were important.

As a simple example the polynomial evaluated at involves, working from left to right and including powers, eight calculations including 5 multiplications and 3 additions, so that Alternatively, using Horner’s scheme, the four part expression for the polynomial is rearranged to show . Thus working from the inside brackets out, we have that and finally . This involved only five calculations, because subtotals are carried forward in each of the three main steps. If we denote, as a linear polynomial, then we can likewise denote and . Hence Horner’s scheme reorganizes into a series of nested linear polynomials.

Taking another example, the polynomial directly evaluated at would involve no less than 14 mathematical operations whereas Horner’s scheme of , , and finally not only has just calculations, but is almost rhythmical in execution. Moving from left to right, each subtotal is formed by multiplying the chosen value by the previous subtotals (with the leading coefficient acting as the first subtotal) and then adding on, or subtracting from, depending on its sign, the next coefficient in line. A 19*th* century engineer, practiced in the technique, might only need to remember the subtotal found at each new stage of the calculation and mentally record and 129.

**Lill’s realisation**

Consider the general cubic polynomial given by The Horner scheme reorganises this to , which, using and , can be neatly expressed as .

Horner’s arrangement minimizes the number of multiplication and addition operations in evaluating but finding the zeros of is a different problem.

Perhaps Lill saw the nested linear structure in Horner’s expression and imagined putting and then , so that if , three interlinked equations emerge:

Clearly, the ratios on the right are all equal, suggesting the analogy with similar triangles. We might note in passing that it is the *ratios* between the coefficients in a polynomial (not their absolute values) that determine its zeros.

Lill must have imagined triangles like the following to match the equations. By making the triangles right-angled, the number would be a tangent.

Rearranging the triangles, Lill would have obtained

The coefficients of the polynomial are now arranged as orthogonal line segments containing a rectangular path. There is a clear relation between the gradient of the hypotenuse of the lower triangle and a zero of the polynomial.

Note that, for any polynomial , if then is a root of the polynomial. Thus if we let then a root of is given by .

Lill then reverse engineers this relationship. He supposes that the outer edges of the above diagram represent the polynomial and its coefficients, while the inner hypotenuses represent a sort of strange path (a light ray perhaps) emanating from a point and striking these outer edges and reflecting improperly to create two inner right angles before reaching its terminal point .

The root of the polynomial is now just the negative of the gradient of that part of the ray that first ‘strikes’ the outer vertical line. In other words, is a root of .

An engineer using the technique would simply draw the outer lines according to the coefficients of the polynomial, and then try to determine the path of a ray that would, after reflecting strangely off the outer lines, terminate at the point . This would take some doing, but when it was done, the negative of the ratio would be a root of .

**The general case**

The forgoing discussion has focused on the cubic equation, but Lill was able to demonstrate that the technique worked for polynomials in general. The diagrams above relating to the general cubic polynomial assumed that every coefficient was positive, but quite obviously this is not always the case. So a methodology had to be developed for depicting all cases of all polynomials. Lill realised that the similarity of the right-angled triangles was the key to any generalisation, and so set about devising a set of rules that would allow as many similar triangles to connect with each other as is needed to allow the connection of to .

We will meet a variety of examples of Lill diagram representation of various degree polynomials shortly, but for now consider this representation of a certain quadratic polynomial

The three solid black lines represent the polynomial, and the reason it looks different to the ones we have so far encountered is that there are negative coefficients involved. Getting from to is not possible directly, because a ray will be reflected to the lower left as soon as it strikes the vertical line segment of unit length. So the diagram shows two rays, one red and one blue, and the red ray emanates from in a northeasterly direction. The vertical line, when extended intercepts this ray, and in order to ensure that similar triangles connect onto Lill permits an ‘orthogonal refraction’ rule through the extended line. The geometry is exactly the same

The critical observation is the linking of the hypotenuses of similar triangles from where the ray starts to where it finishes. The same applies to the other blue coloured path. This quadratic equation happens to have two symmetric roots.

Lill developed a set of rules to aid practitioners who perhaps were not always familiar with Euclidean geometry concepts. These rules specified how to construct line diagrams for any given polynomial and how to draw the zigzagging ray lines reflecting and refracting in ways completely foreign to natural laws of physics. The problem with algorithms like this is that following them blindly without really knowing the beautiful mathematics that underpins them seems like witchcraft.

In researching this work, the authors were able to find many online sites that spelt out Lill’s algorithm well, but in most cases no mathematical explanation was forthcoming. Many people who had contributed to chat forums seemed completely at a loss as to how Lille came up with the rules in the first place. This perhaps is a useful metaphor for poor pedagogy where too much emphasis and value is placed on the memorising of certain mathematical rules without the mathematics to justify them.

In the examples that follow, you will see diagrams constructed from a set of rules that Lill devised, but you will also notice the similar triangles that are necessary to form the links. These triangles not only visually explain the rules but also highlight the wisdom and genius of Lill himself.

**The Lill diagram for a quadratic polynomial**

Lill’s method, as it is known, is perhaps best described by a number of examples. We begin with the quadratic polynomials.

1. **The roots of**

We begin with an origin

We note that there are three coefficients in the quadratic, namely and and so we need three line segments. Lill’s idea is to imagine each segment as a section of path that, say, a robot is walking along. The direction it walks and the direction it faces is critical to drawing an accurate path. Because the first coefficient is imagine the robot walking from in an easterly direction as shown.

When the robot finishes any section of path, its final action will be to rotate a quarter turn anticlockwise. In this instance, the robot’s action will be to rotate to North, so we shall indicate this by showing a direction-facing symbol like this

The second coefficient is units, so in Lill’s procedure, the robot walks backward units. The negative coefficient is interpreted to mean that the robot walks backward from the direction they are facing.

Note that the robot has moved backwards and then changed direction to face west.

The third coefficient is positive and so the robot walks forward 6 units.

The representation of the polynomial is now complete, and all that is left to do is to locate the correct angle from which a particular ray leaves , reflects off the middle line segment and meets .

We’ll find two possible paths, because there are two real roots of the polynomial. We have cheated by using algebra to find them, but strictly speaking Lill’s method often involved guessing and adjusting iteratively, particularly when the degree of the polynomial was 3 or more. The two possible paths are shown here in red and blue

The gradients of the first segments are respectively and and so the roots are and . The geometry is clear. With respect to the red ray, the two similar triangles are shown here

The ratio of sides, and of the two highlighted right-angle triangles are equal and the root of the polynomial is the negative of the gradient in the smaller triangle. The same idea applies to the other ray.

The Horner scheme for the polynomial becomes .

**2. The roots of**

For the polynomial we need to remember that even though the coefficient of the linear term is zero, the robot still must complete its anticlockwise quarter turn. We write and proceed with drawing the Lill diagram. From the robot walks eastward unit and then rotates to the north. Then walks a distance of zero, and rotates anticlockwise again. The robot is now facing west. Finally because of the constant term, the robot now moves backward 16 units. The representation of the polynomial is a straight line.

In the process of turning north momentarily Lill’s rules allow us to extend the zero length line vertically in both directions, and it is this extension that enables the construction of two rays shown here. Lill allows a refraction procedure as shown, and because we know the roots are the rays need to cross the extended line 4 units above and below the centre line and as they do alter their direction by so that they each meet at .

It’s not hard to spot the triangles that span and . The two smaller right angle triangles to the left of the extended line have and gradients of and respectively. The two roots are thus and . The two larger rectangles on the right side of the extended line have their shorter sides in the same ratio.

**3. The roots of**

Going though the same step-by-step process with the coefficients and we construct the Lill diagram and rays as follows. Note that the robot moves 1 unit to the right and turns north, then moves backward 1 unit and turns west, and then reverses again 1 unit and turns south.

In this case it turns out that where is the golden ratio.

This is true because, from similar triangles in the figure, we have that or that or , the positive root of which is given by . Once again, the triangles span and like every other example we have met.

Note that the geometric mean of and is 1 and this can be readily seen as the altitude of the largest right angle triangle on each side of the vertical line.

**4. The roots of**

With practice the construction of the polynomial representation is easily followed. This

Since is the perfect square , only one ray is possible.

**5. The single root of**

For the linear polynomial, there is a direct path from to so that the gradient . The single root of the polynomial is .

**6. The roots of**

The polynomial factorises easily to the product of three distinct linear factors so that . The Lill diagram and the three rays from to are shown here

Take particular attention to the three connecting similar triangles from to for each ray shown. As an example, focusing on the blue ray, the ratio of the shorter sides of the first and smallest right-angle triangle is . For the next largest similar triangle made by the blue ray, we can find the ratio by solving from which . Thus the final similar triangle has its two shorter sides as and .

You’ll notice also that the ray arrows have not been put in, and this is to highlight another property of these diagrams. Suppose we swap the positions of and , and then rotate and flip the entire representation. What polynomial are we describing?

Appealing to our analogy, the robot has now moved forward 6 units, reversed 11 units, moved forward another 6 units, and finally reversed one unit. This corresponds to the new polynomial factored as .

The operation of rotating and reflecting has the effect of interchanging the constants in each linear factor. Thus we have . That is to say the roots are reciprocated as implied by the reoriented triangles.

**7. The roots of**

The equation, when written , alerts us to the need to include two quarter turns despite the robot not moving laterally.

2

**A square boundary**

The ratio of successive terms of the Fibonacci series … given by is known to approach the golden ratio Perhaps not as well known is that the analogous ratio for the tribonacci series given by where each term is the sum of the previous three terms, also approaches a limit . In fact the ratio of terms of the series where each term after the first ‘ones’ is formed by adding the previous terms approaches a limit .

It can be shown that the term is the single positive root of the polynomial equation given by . An example of this is the golden ratio derived as the positive root of the equation . Of course the root of is .

The Lill diagrams for the set of equations of the form , for , are, apart from the location of the terminal point , identical as shown here

For , relating to the quadratic equation , the Lill diagram follows the path where the terminal is the point . For the path becomes and for the path completes the square following .

As increases, the path continues around and around the square terminating at one of the four possible stops.

To draw the rays associated with this diagram we need to include the extensions of these lines in every direction as shown on the next page.

As an example, the rays for solutions to , , and have been drawn in. Notice that their rays form a cyclic pattern, so that the ray associated with changes direction times before it connects with a corner of the square. Using a spreadsheet we can obtain estimates of the phi constants as lengths along the extension of as , and etc.

For the general - generating polynomial note that can never be a solution, for upon substitution we have the expression for every integer value of . This implies a limit exists where . Thus, in the sequence none of the ratios will equal and so the no ray will go beyond an outer square whose sides have gradients and .

In his popular book Mathematical Puzzles and Diversions, Martin Gardner notes that Mark Barr, the person who gave phi ( its name, may have been the first to generalise the phi concept. In a newspaper article (*The London Sketch* 1913) Barr states that and are related by the equation

So for example when , the Fibonacci sequence, is the familiar golden ratio. For etc. The reader can check other values.

A rearrangement of this interesting equation reveals that the polynomial given by is at the heart of this relationship, and we can construct Lill diagrams for various to locate the positive solutions to .

The Lill diagram for is shown here

**The ray – a scaled polynomial in disguise**

Let us now return to the polynomial discussed in example 6, shown here with just one of the three rays drawn in.

2

2

The ray, formed by the hypotenuses of three similar triangles, identifies the root . The length of each section of the ray has been identified in the diagram. Dividing by the factor yields , a quadratic polynomial that has the factorized form .

The representation of as a Lill diagram is shown below, and it is of some significance that the ratios of the length of each segment is exactly the same as the ratios of lengths of the ray segments above. In other words, the representation of is a scaled down version of the ray, (the scaling factor is , the length of the hypotenuse of the highlighted triangle).

If we repeat the exercise of finding a suitable ray for , we find that the ray itself is a scaled representation of the linear polynomial . This time the scaling factor is .

In other words, the rays themselves are scaled versions of polynomials that are one degree lower than the polynomial representation they sit in.

Because the ratios remain unchanged, the scaled polynomials can be visually represented together on the one diagram as shown here.