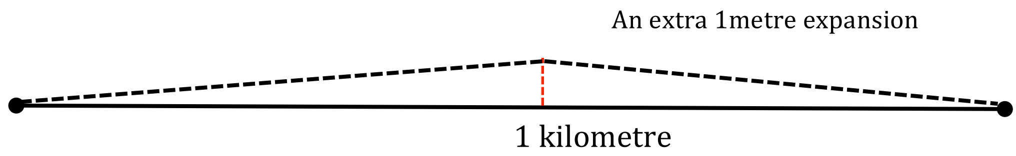
21 Puzzles

Ed Staples & Paul Turner
2019

Puzzle 1: The expanding railway track

A straight piece of railway line one kilometer long and fixed at both ends expands by a single metre from the heat causing it to buckle at the centre as shown in the diagram (not to scale).

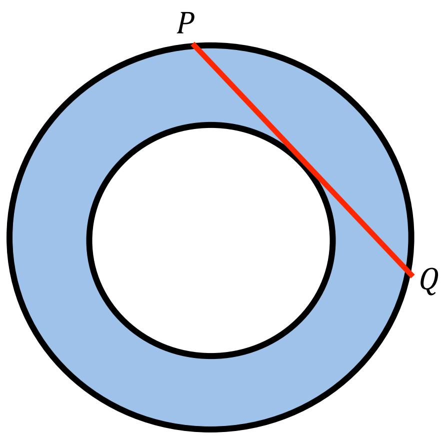


Side view of track

How large do you imagine the gap (shown in red) that appears at the centre? Could an ant, a mouse, or even some larger animal get under it?

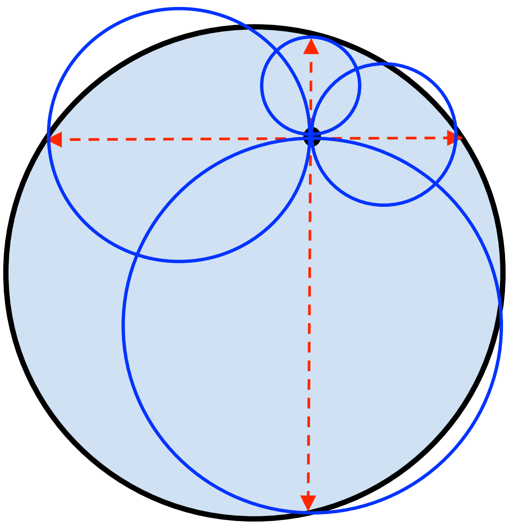
Puzzle 2: The water fountain

A pond in the form of an annulus consists of two concentric circles as shown in the diagram. The length of PQ is 12 metres. Find the surface area of the water.



Puzzle 3: Four circles

A point is chosen at random inside a circle and two orthogonal chords are drawn through it as shown in the diagram. These chords divide each other into four line segments (none of which need be equal to any other). Four circles have been drawn with diameters as these line segments. It's a remarkable fact that the combined area of the four circles is equal to the area of the original circle. Can you prove it?



Puzzle 4: Sharing the profit

Sam and Sandra find themselves at adjacent stalls of the Ballarat markets, held on the first Sunday of every month. They each have 30 daffodil bulbs sourced from the same supplier, but Sam has his at two for a dollar and Sandra has hers at three for a dollar. They decide it is silly to be in competition with each other, and agree to combine their 60 bulbs and sell them for five for two dollars.
They sell all of the bulbs that day, but are confused about their profit. If Sam had have sold his at two for a dollar, he would have made $15. Similarly Sandra would have made $10 for her 30 bulbs to take the total to $25. But the combined profit on 60 bulbs at five for two dollars was only $24.
Where did the other dollar go?

Puzzle 5: The Inheritance

Three brothers Adam, Bill and Charles had recently learnt that they were to receive an inheritance from a dairy farmer well known to the family. His gift was 17 dairy cows and the will was very specific as to how the cows were to be disbursed among the brothers. Adam was to receive half of them, Bill a third of them, and Charles a ninth of them.

Not wanting to kill any of the cows, the boys sought advice from their wise uncle who happened to own a dairy cow himself. "Simple" said the uncle; "I'll give you my cow, so that will bring the total number of cows to 18.

In accordance with the will, Adam will then receive 9 cows, Bill will receive 6 cows and Charles will receive 2 cows. That means 17 cows all up so I then can take my cow back, unharmed!" Adam was baffled by the simplicity of the uncle's solution but suspected that something was not quite right. What do you think?

Puzzle 6: The HCF and LCM

"I've forgotten the two whole numbers I wrote down", said Simon, "but I know that the product of their highest common factor and their least common multiple was 108. Their sum was a two-digit prime as I recall, but I'm afraid that's all I remember."

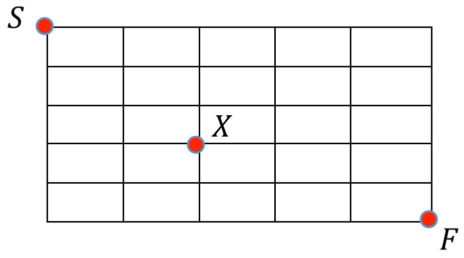
Can you help Simon?

Puzzle 7: The Cricket team

A cricket team consists of 11 players. At the beginning of a game of cricket they make their way, usually in single file, onto the cricket ground. How many different orders can the players line up in before coming on?

Puzzle 8: The Ant and its journey

An ant must travel from S to F along the grid lines by going either East (E) or South (S) only. The only other restriction is that it must visit the point X on its journey. How many ways can the ant complete the journey?



Puzzle 9: The locker problem

The East wing corridor of St Anne's School of Hard Knox has 100 senior school lockers consecutively numbered from 1 to 100. Student 1 opens up every locker and leaves. Student 2 closes every second locker (lockers 2, 4, 6...100). Student 3 changes the state of every third locker by closing it if she finds it open or else opening it if she finds it closed. This pattern of opening and closing continues with student n changing the state of every nth locker up to student 100 who changes the state of the 100th locker. Which lockers remain open and which lockers remained closed?

Puzzle 10: The average speed

A cyclist rides his bike at an average speed of 20 km per hour from place A to place B. Without stopping to rest he rides back to A at 10 km per hour. What was his average speed for the journey?

Puzzle 11: Alphabet multiplication

If A, B, C and D are the four different digits of a number $ABCD$, and if we know that $4 \times ABCD = DCBA$ then the value of the digits can be determined by using a little bit of logic.
The best way to start thinking about the problem is to set the product up as if multiplying without a calculator. Can you find the values?

$$\begin{array}{cccc} A & B & C & D \\ & & & 4 \\ \hline D & C & B & A \end{array}$$

Puzzle 12: Magic Products

The trick of creating magic squares of odd order is well known. For example, with the order 5 magic square, start at the bottom centre with the number 1 and then work up and right until you reach the edge of the square with 3. Then wrap around and up to 4 and then 5 (shown in red). The first diagonal line is then complete. Then start a new diagonal, wrapping where required, directly underneath where the last diagonal finished (shown in green). Continue filling in the diagonals until the square is full.

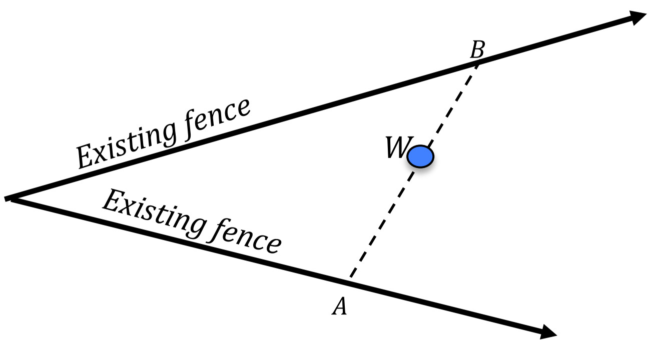
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9
17	24	1	8	15

The sum of the columns, rows and two main diagonals all add to the magic number 65.

The trick of creating a magic product square, where all rows, columns and diagonals add to the same magic product, is not as well known. It's not that difficult if you think about it. Can you create one?

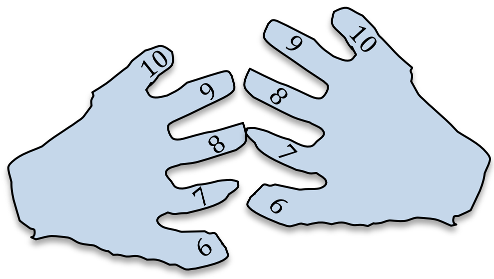
Puzzle 13: The smallest enclosure

A farmer wishes to fence off the smallest triangular area possible by using one new straight line of fence *AB* that would have to intersect an immovable water trough to join two other existing fences as shown in the diagram. The trouble is she is not sure of the direction the fence should go to ensure the enclosure is minimized. Is there a way of determining where to build the new fence?



Puzzle 14: Finger multiplication

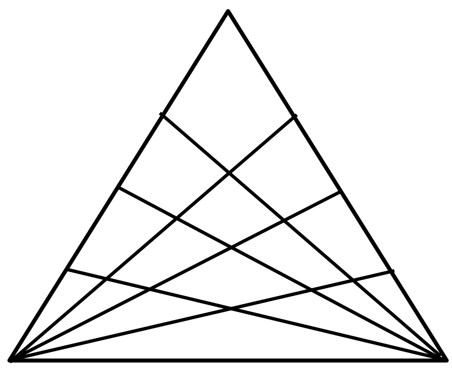
A clever little trick using your hands exists to multiply two numbers whenever they are both between 6 and 10.



The diagram shows how to multiply 8 by 7. Each finger is numbered as shown, and the two fingers touching, the 8 and 7, become the 'bridge'. There are two steps to finding the product. Count the number of fingers below and including the bridge. There are 5 of these, and these count as tens, so together they make 50. Above the bridge there are 2 fingers to the left and 3 fingers to the right, so multiply 2 by 3 to give 6. Add the two answers to 56, the desired product. Why does this work in every case?

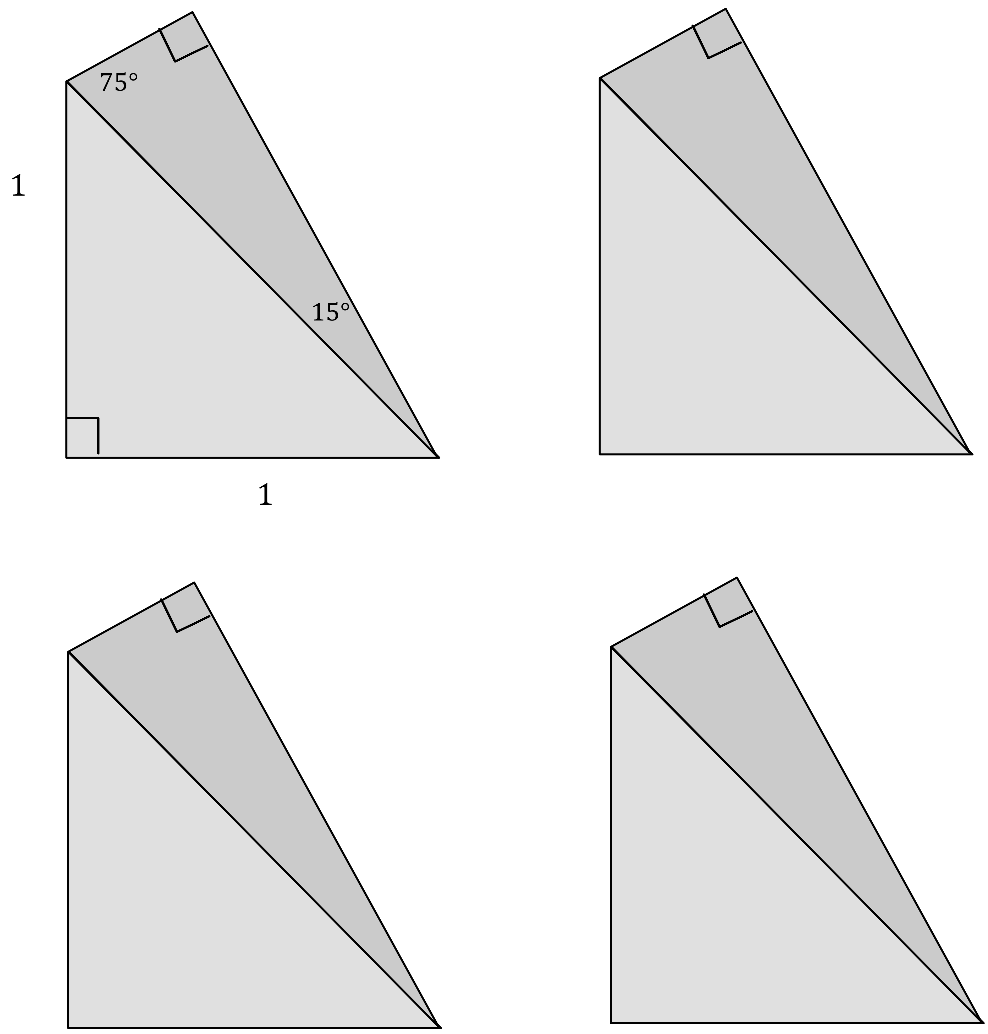
Puzzle 15: Counting triangles

How many triangles are there in this figure?



Puzzle 16: Versa-tiles

Print off and cut out these four identical shapes.



1. Can you find the exact length of the two unknown sides of the figure and thereby determine the area of each piece?

2. Can you use the four shapes to make a square with a square hole in it? What is the exact area of the square hole?

3. Can you rearrange the shapes to make a square and a parallelogram that doesn't contain any holes?

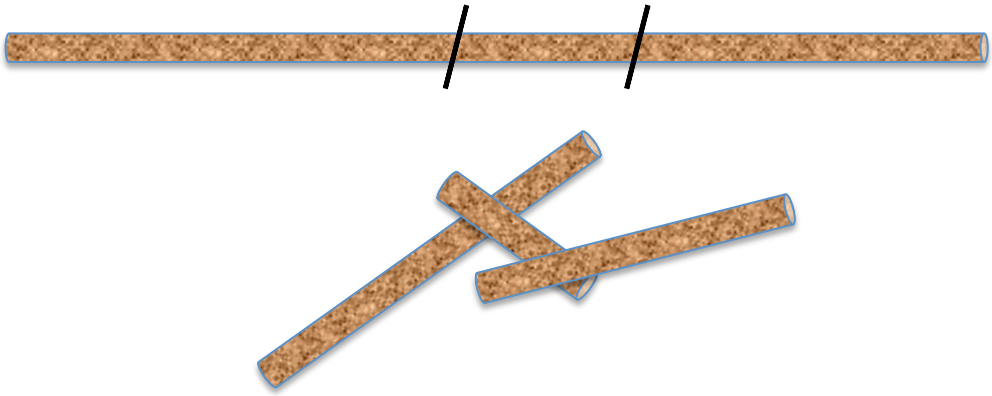
4. Can you rearrange three of the shapes to make an equilateral triangle with a hole in it? What is the exact area of the hole?

Puzzle 17: Two boys

Jane has two children and one of them is a boy. What is the probability that the other child is a boy also?

Puzzle 18: From sticks to triangles

A stick is broken randomly in two places and the three parts form a triangle. Will this always happen? If not then what is the probability of the pieces forming a triangle? Does the answer change depending on how the stick is broken?



Puzzle 19: Sums of squares

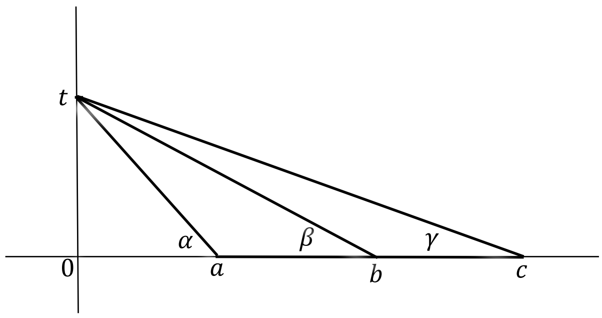
The number 58 can be expressed as the sum of the squares of 7 and 3. The number 29 can be expressed as the sum of the squares of 5 and 2. This wouldn't be at all interesting if it was not for the fact that half the sum of 7 and 3 is 5 and half the difference of 7 and 3 is 2, and half of 58 is 29. Is it true that in general if the number $2n$ can be expressed as the sum of squares of x and y then n can be expressed as the sum of squares of u and v where u is half the sum of x and y and v is half the difference of x and y?

Puzzle 20: Gardner triples

A geometrical discovery made by Paul Turner and Ed Staples in 2019 is described as follows:

1. Pick any two relatively prime integers a and t so that $t < a$
2. Find $S = a^2 + t^2$
3. Choose any divisor pair (d_1, d_2) such that $d_1 \times d_2 = S$
4. Set $b = a + d_1$ and $c = a + d_2$
5. Then $\tan^{-1}\frac{t}{a} = \tan^{-1}\frac{t}{b} + \tan^{-1}\frac{t}{c}$

Turner and Staples referred to the relationship as the *Type t Gardner Triple* $(a, b, c)_t$ (named after the popular mathematics writer Martin Gardner) and the result can be depicted geometrically where $\alpha = \beta + \gamma$.



How might have Turner and Staples derived this result?

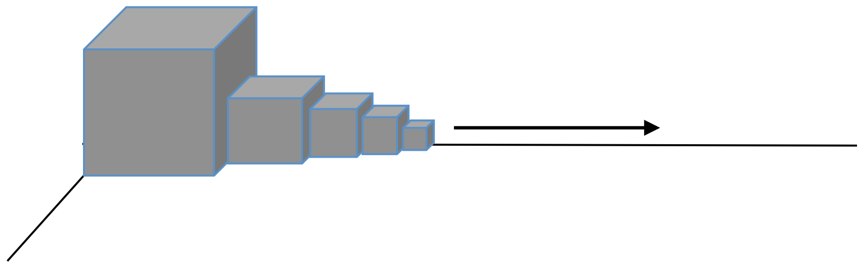
Use the Turner/Staples algorithm to find a Gardner triple when $a = 3$ and $t = 2$.

Can you verify the existence of the following Fibonacci related type 1 Gardner triples $(1,2,3)_1$,$(3,5,8)_1$, $(8,13,21)_1$, $(21,34,55)_1$?

For more information on Gardner triples see the series of articles published in the Australian Mathematics Education Journal published by the Australian Association of Mathematics Teachers in 2019.

Puzzle 21: Euler's boxes

Dr. Euler asked us over to look at some of his cardboard boxes. He had made ten of them and had pushed them together in a line against the 15 metre back wall of the University gymnasium. Not one of the boxes was the same size as any other. It was the oddest thing to look at because they became so small after a while.



He explained that they were all cubes with the largest, at the left corner, having a side of exactly 1 metre. The box next to that had a side of $\frac{1}{2}$ metre, and the one after that $\frac{1}{3}$ of a metre, and after that $\frac{1}{4}$ metre, and so on with the nth box having a side of $\frac{1}{n}$ of a metre. The first five are shown in the sketch but I counted five more and the line of boxes was heading toward the right hand corner of the gym.

I enquired about how many boxes it would take to get to the right hand corner of the gymnasium and he immediately replied that it would take 1,835,421 boxes to cover the 15 metres. He said that he knew that a sufficient number of boxes could make the line of boxes as long as one wished.

This sort of made sense to me but then he said something quite strange. He told me that even though he could theoretically make the line of boxes as long as he wished the total amount of cardboard used to build them would never exceed 10 square metres. This had me rattled for it seemed he used most of that cardboard on the first five boxes!

How much cardboard was used to make the first five boxes? Why could Dr. Euler declare that the line could be made as long was anyone wished? How did he know that, at least theoretically, the total amount of cardboard used to build them would never exceed 10 square metres?

Solutions

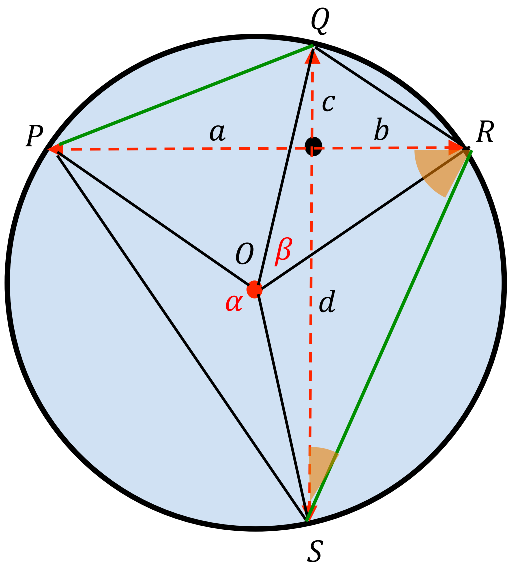
Puzzle 1: The expanding railway track

Using Pythagoras' theorem we solve for h the height of the gap.
Thus, working in metres, we have $h^2 = 500.5^2 - 500^2 = 500.25$ and so, remarkably, the gap becomes about 22.37 metres. This means that any animal would be able to walk through the gap.

Puzzle 2: The water fountain

This question is interesting because the radii of the two circles are not given and yet the question demands an answer. This can only mean that the answer is independent of these radii. Suppose that we imagine the inner circle's radius to be extremely close to 0 metres (note that we must keep the pond as an annulus). Then the surface area of the water must be approximately the area of the larger circle, with PQ approximately a diameter. In the limiting case it stands to reason that the area of the annulus must be given by $A = \pi r^2 = \pi \times 6^2 = 36\pi$.
If we introduce algebra and solve for the area of the general annulus we arrive at the same value.

Puzzle 3: Four circles



The length of PS can be found by Pythagoras' Theorem and the cosine rule, so that

$$PS^2 = 2r^2(1 - \cos\alpha) = a^2 + b^2$$

Similarly we have, in respect of QR,
$$QR^2 = 2r^2(1 - \cos\beta) = c^2 + d^2$$

Thus $a^2 + b^2 + c^2 + d^2 = 2r^2(2 - \cos\alpha - \cos\beta)$ and multiplying both sides of this equation by $\frac{\pi}{4}$ shows

$$\frac{\pi}{4}[a^2 + b^2 + c^2 + d^2] = \pi r^2 \left[1 - \frac{1}{2}(\cos\alpha + \cos\beta)\right]$$

Looking back at the diagram we know that $\angle PRS$ is $\frac{1}{2}\alpha$ and $\angle RSQ$ is $\frac{1}{2}\beta$ (because the angle at the centre is twice the angle at the circumference subtended by the same chord). But these two angles are complementary angles, implying that $\alpha + \beta = 180$, and consequently that $\cos\alpha + \cos\beta = 0$.

Thus $A = \pi r^2 = \frac{\pi}{4}[a^2 + b^2 + c^2 + d^2] = \pi\left(\frac{a}{2}\right)^2 + \pi\left(\frac{b}{2}\right)^2 + \pi\left(\frac{c}{2}\right)^2 + \pi\left(\frac{d}{2}\right)^2$

Since the point in the circle was chosen at random this is a completely general result. It remains true when the point is on the circumference, and can be shown to be true if the point lies within the circumscribing square. For further reading see https://files.eric.ed.gov/fulltext/EJ853816.pdf

Puzzle 4: Sharing the profit

Sam sells his bulbs at two for a dollar, or 50 cents each. Sandra sells hers at 3 for a dollar or $33\frac{1}{3}$ cents each. When Sam and Sandra combine to sell five for two dollars, they sell them for 40 cents each. Note that

$$30\times0.50 + 30\times0.\dot{3} = 30\times(0.5 + 0.\dot{3}) = \$25 \neq 60\times0.40 = \$24$$

Algebraically, if Sam sells at m for a dollar and Sandra sells at n for a dollar the total profit would be $\frac{30}{m} + \frac{30}{n}$. Combining at $(m + n)$ for two dollars delivers a profit of $\frac{120}{m+n}$. Setting $\frac{30}{m} + \frac{30}{n} = \frac{120}{m+n}$ shows that the two situations can only be equal when $m = n$.

Puzzle 5: The Inheritance

Strictly in keeping with the farmer's wishes, Adam was due to receive $8\frac{1}{2}$ cows, Bill was due to receive $5\frac{2}{3}$ cows and Charles was due to receive $1\frac{8}{9}$ cows. Based on the farmer's fractions the total allocation was therefore $16\frac{1}{18}$ cows, so that $\frac{17}{18}$ of a cow remained unallocated. Sadly, it also meant the demise of 3 cows.

However, if we preserve the relative ratios between the brothers, and allocate all 17 cows (which perhaps could have been the farmer's original intent) we find that Adam, Bill and Charles should have received the proportions $\frac{9}{17}, \frac{6}{17}$, and $\frac{2}{17}$ respectively anyway.

Puzzle 6: The HCF and LCM

Hidden within this question is a fascinating result of number theory.

Every whole number N can be represented as a product of multiplicities of the primes. In general then we have $N = 2^{\alpha}\times3^{\beta}\times5^{\gamma}\times7^{\delta}\times$ ... where the exponents are non-negative integers (including 0).

To illustrate the general result first consider the two numbers given by $24 = 2^3\times3^1\times5^0\times7^0\times \ldots$ and $63 = 2^0\times3^2\times5^0\times7^1\times$ ... and their product $1512 = 2^3\times3^3\times5^0\times7^1\times \ldots$.

The prime factors in the least common multiple of these two numbers must contain exactly three 2s, two 3s and a 7. For example, if it only contained two 2s, then it would not be a multiple of 24 because 24 contains three 2's. If it only contained one 3, then it could not be a multiple of 63 which contains two 3s. In other words, the least common multiple of 24 and 63 must contain the *maximum* number of factors of each prime present in either number. Hence the least common multiple is $2^3\times3^2\times7 = 504$.

In a similar way, the prime factors in the highest common factor of the two numbers must contain the *minimum* number of factors of each prime present in either number. This means that the highest common factor of 24 and 63 must be $2^0\times3^1\times7^0 = 3$.

The product of the least common multiple and the highest common factor becomes$(2^3\times3^2\times7)\times(2^0\times3^1\times7^0) = 2^3\times3^3\times7^1 = 1512$, which can only be the product of our two numbers.

In respect of Simon's numbers, he remembers that the product of their highest common factor and least common multiple is 108. This means that the product of his two numbers must also 108, and so the two numbers must be one of the pairs $(1, 108)$, $(2, 54)$, $(3, 36)$, $(4, 27)$, $(6, 16)$ and $(9, 12)$. Thus, because the sum is a two-digit prime, Simon's numbers must be 4 and 27.

Puzzle 7: The Cricket team

In most counting problems in mathematics, the fundamental counting theorem comes into play. If there are m ways of completing task A and n ways of completing task B then there are mn ways of completing both tasks. This is true because for every single way task A can be completed there are n ways that B can be completed, and thus multiplication makes sense in the principle.
If there are more than two tasks then the ways each task can be completed are simply multiplied together.

For our cricket team any one of the 11 cricketers could walk onto the ground first. After that, any of the remaining 10 cricketers can walk onto the ground second. So there are 11×10 choices available for the selection of the first two cricketers. It follows that there are $11 \times 10 \times 9 \times 8 \times \ldots \times 3 \times 2 \times 1 = 39{,}916{,}800$ ways that 11 cricketers could walk onto the ground. Mathematicians abbreviate this product as 11! and call it 11 factorial.

The size of 11! is staggering. Whether it is cricketers coming onto the field or horses passing the winning post or line-ups for a group photograph there are 39,916,800 ways to do it. The sizes of these factorials have been mischievously used to disguise the odds of winning on many betting games, and this includes horse racing, poker machines and lotto games.

For example, there are an enormous number of ways of selecting six numbered balls from a barrel of 45 balls. Based on factorial arguments like the one above it can be shown that there are 8,145,060 ways of making the selection. There are $45 \times 44 \times 43 \times 42 \times 41 \times 40 = 5{,}864{,}443{,}200$ orderings of six balls that can be made but many of these include the same balls arranged in different orders. There are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ orderings of any particular set of six balls, and so when these are divided out, the number of different combinations of six balls reduces to 8,145,060.

Not understanding factorials can cause a player to be misled on their chance of winning. The illusion of reasonableness when picking six balls from forty-five balls is why the game survives. For example, if a weekly selection of one ball from 8,145,060 balls (each of which had a particular combination of six numbers written on it) occurred then most players would think twice about participating in the game.

Puzzle 8: The Ant and its journey

Any journey the ant makes must contain five moves to the east and 5 moves to the south. That's five Es and five Ss in any order. Hence without the X restriction there must be $\frac{10!}{5! \times 5!} = 252$ different journeys.
The restriction of X is easily handled by breaking up the trip into two sections. From S to X there are $\frac{5!}{2! \times 3!}$ different journeys and similarly from X to F there are also $\frac{5!}{3! \times 2!}$ journeys. Together there must be $\frac{5!}{2! \times 3!} \times \frac{5!}{2! \times 3!} = 100$ ways for the ant to traverse the grid.

Puzzle 9: The locker problem

The key to this beautiful question is to understand that most lockers are visited an even number of times and a few are visited an odd number of times. When a locker is visited an even number of times, its final state will be closed. When a locker is visited an odd number of times its final state will be open.

For example students 1, 2, 4, 5, 10 and 20 are the only students that can visit locker 20 and these numbers happen to be the 6 factors of 20. Students 1, 5, and 25 are the only three who will visit locker 25 and again 1, 5 and 25 are the three factors of 25.

In fact the only numbers that have an odd number of factors are square numbers like 1, 4, 9, 16, 25, etc. All other numbers have factors that can be grouped completely into factor pairs. In respect of the number 20 for example the factor pairs are (1,20), (2,10) and (4, 5). Lockers whose numbers are square numbers will have their state changed an odd number of times and since they started closed, they finish open. All other lockers will be visited an even number of times and thus their state will be in the state that they started in, namely closed.

Puzzle 10: The average speed

Imagine the distance from A to B as d km.

The speed of the forward journey to A was 20 km per hour and thus the time taken to A is given $t_A = \frac{d}{20}$. Similarly the time taken to get back to B is $t_B = \frac{d}{10}$.

The total time for the distance $2d$ is therefore given by $t_A + t_B = \frac{d}{20} + \frac{d}{10} = \frac{3d}{20}$ and this means that the average speed for the entire journey becomes $\frac{2d}{\left(\frac{3d}{20}\right)} = \frac{40}{3}$ or a little over 13 km per hour.

The average speed is not the arithmetic average of 15 km per hour because more time is spent on the return journey than the forward journey.

In general with speeds v_1 and v_2 the average speed of the journey is given by the harmonic mean of v_1 and v_2 viz. $V_{average} = \frac{2v_1v_2}{v_1+v_2}$.

Puzzle 11: Alphabet multiplication

We can solve the unknowns progressively. Firstly the number $4 \times A$ cannot exceed 9 or else the product would contain five digits. This means A is restricted to 1 or 2 (note that $A \neq 0$). But A couldn't be 1 because $4 \times D$ must be even, so A must be 2. This means D must be either be 3 or 8 since $4 \times D$ must end in 2. But $4 \times A$ is at least 8 (there could be a carry over of 1 from the previous column) leaving $D = 3$ as the only possibility. Hence there can be no carry over into the fourth column. This in turn restricts B to 1 since 2 is already taken and $4 \times B$ is a single digit. Here are our results so far

$$\begin{array}{r} 2\ 1\ C\ 8 \\ 4 \\ \hline 8\ C\ 1\ 2 \end{array}$$

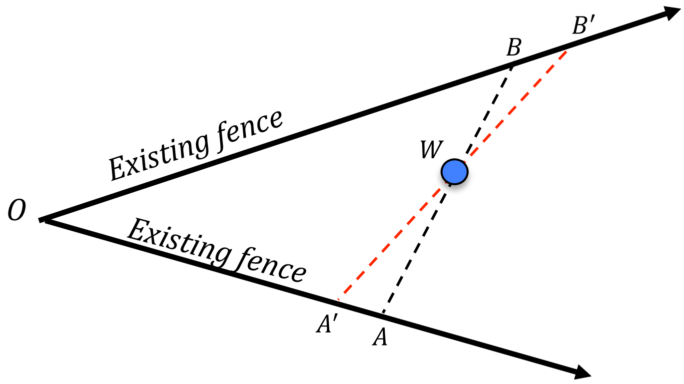
Thus there is a 3 carryover into the second column so $4 \times C + 3$ is a two digit number ending in 1. The only possibility is $C = 7$ since 2 is taken, and so the number $ABCD$ is 2178 and $4 \times 2178 = 8712$.

Puzzle 12: Magic Products

The trick is to imagine the numbers in the magic square to be exponents. Choosing say a base of 2, we can construct a magic product easily.

2^{23}	2^{5}	2^{7}	2^{14}	2^{16}
2^{4}	2^{6}	2^{13}	2^{20}	2^{22}
2^{10}	2^{12}	2^{19}	2^{21}	2^{3}
2^{11}	2^{18}	2^{25}	2^{2}	2^{7}
2^{17}	2^{24}	2^{1}	2^{8}	2^{15}

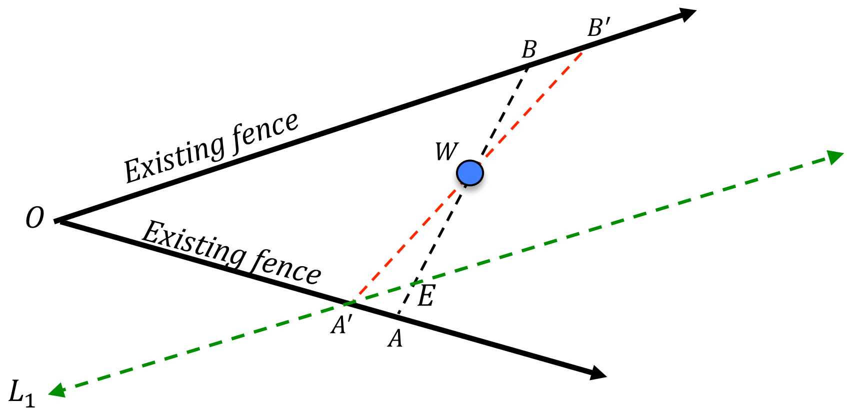
Puzzle 13: The smallest enclosure



Our diagram assumes that W is closer to B than to A so that $\overline{WB} < \overline{AW}$. However, if the situation was reversed, a similar argument to the one that follows could be established.

The question is answered by knowing that a line, say $A'B'$ (shown in red), that makes W a 'midpoint' minimises the area of the enclosure.

To prove this, we first need to construct the line L_1 parallel to one of the existing fences and intersecting AB at E.



Then triangles WBB' and WEA' are congruent (using Angle, Angle, Side) and since E divides WA internally, the area of triangle WBB' is less than the area of triangle WAA'. Hence, rotating the existing line AB clockwise about W to the line $A'B'$ reduces the area of the enclosure by an amount equal to the area of triangle EAA'.

But AB was any line between the existing fences and so this implies that the enclosure defined by $A'B'$ can never be any more than AB. Thus $A'B'$ minimises the area of the enclosure.

Puzzle 14: Finger multiplication

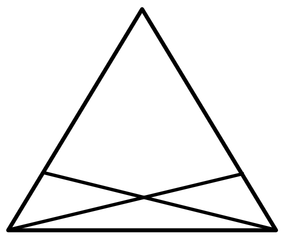
Let the left and right hand fingers correspond to the numbers $5 + m$ and $5 + n$ for values of m and n between 1 and 5. Then, for specific values of m and n, the fingers that correspond to $5 + m$ and $5 + n$ will form the bridge. The total count of fingers below and including the bridge is simply $m + n$.

The count of fingers above the bridge, first left and then right, is $10 - (5 + m\ )$ and $10 - (5 + n\ )$ respectively, and these simplify to $5 - m$ and $5 - n$. Thus, their product becomes $25 - 5(m + n) + mn$.

The claim is that the sought after product is $10(m + n) + [25 - 5(m + n) + mn]$ and this simplifies easily to $25 + 5(m + n) + mn = (5 + m\ )(5 + n)$, which is the very expression we need.

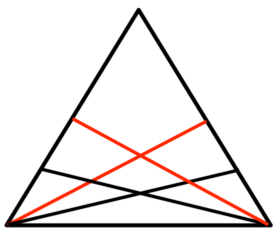
Puzzle 15: Counting triangles

On way forward with this puzzle is to think about how the diagram was constructed in the first place. There a three line segments emanating from each of the two base vertices and going to the opposite sides of the triangle.
We might then begin by considering the case where only one line segment is drawn from each base vertex.



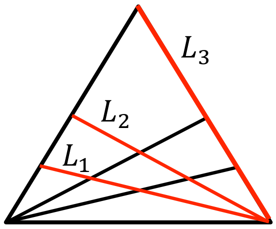
A careful count of the triangles in this figure reveals 8 triangles (1 large triangle, 3 small triangles and 4 triangles that are formed by combining 4 pairs of triangles).

Adding a second pair of lines we find more triangles but it's harder to identify and count them.

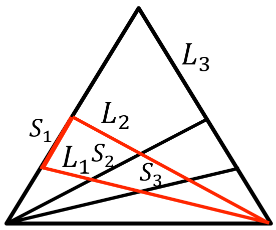


We need another way to think about the problem

Every triangle has three sides and there is a subset of triangles in the figure that use the base of the large triangle as one of their sides. In these triangles one of their other sides could be the negatively sloping red lines those shown as L_1, L_2 and L_3. Likewise their third side could be the three positively sloping black lines. Thus there are $3 \times 3 = 9$ possible triangles in this subset of triangles.



But there are other triangles that do not include the base line as one of their sides. For example, there is a triangle made of the sides L_1 and L_2 and the left most segment S_1 in between L_1 and L_2 as shown here



From the three lines L_1, L_2 and L_3 we need to choose combinations of two of them (note that the base line is not included because we have already dealt with it in the previous subset) and from counting principles there are $\frac{3!}{2! \times 1!} = 3$ pairs formable. For each of these pairs there is a choice of three line segments S_1, S_2 and S_3 for the third side, so that means in total there are $3 \times 3 = 9$ triangles in the figure that have two of their sides sloping downward.

The same would be true if we had used two upward sloping sides, so that doubles the number of triangles to 18.

In total then, including the subset of triangles where the base was used, there are 27 triangles in the figure. So our first two answers are 8 and 27 and both of these are cubes.

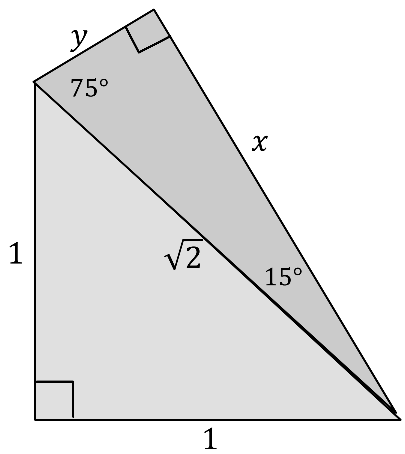
Using the same logic for the actual question that was asked, we would immediately conclude that there would be $(4 \times 4) + 2\left(4 \times \frac{4!}{2! \times 2!}\right) = 64$ triangles.

It is enticing to generalise this result to n lines considering our suspicion of cubes as solutions.

With n lines drawn from each vertex there must be $(n \times n) + 2\left[n \times \frac{n!}{2! \times (n-2)!}\right]$ triangles in the n line figure.

The expression simplifies as $n^2 + [n \times n(n-1)] = n^3$ triangles and our suspicions were correct.

Puzzle 16: Versa-tiles

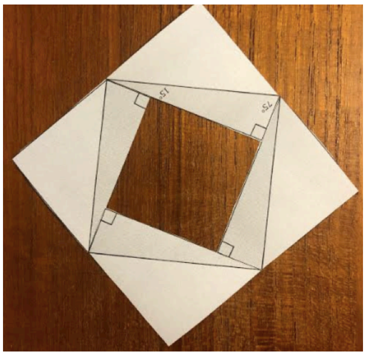


Q1

The length of the hypotenuse of the triangle is $\sqrt{2}$ and so by using some basic trigonometry we have that $x = \sqrt{2}\cos 15°$ and $y = \sqrt{2}\sin 15°$.

Also, because $\cos 15° = \cos 60° \cos 45° + \sin 60° \sin 45° = \frac{\sqrt{3}+1}{2\sqrt{2}}$ and likewise because $\sin 15° = \sin 60° \cos 45° - \sin 60° \sin 45° = \frac{\sqrt{3}-1}{2\sqrt{2}}$, then we find exact values of x and y as $\frac{\sqrt{3}+1}{2}$ and $\frac{\sqrt{3}-1}{2}$. This means that a single piece has a total area of $\frac{1}{2} + \frac{1}{2}\left(\frac{\sqrt{3}+1}{2}\right)\left(\frac{\sqrt{3}-1}{2}\right) = \frac{3}{4}$ square unit.

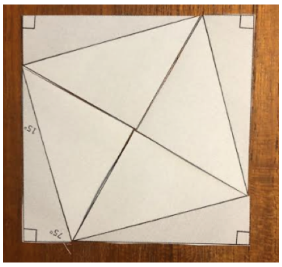
Q2

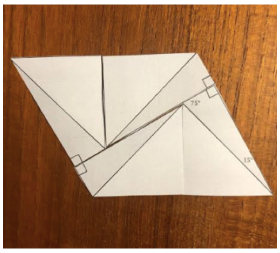


Your arrangement should look like this. Note that the outside square is of side 2 units and the inside square has a side length $\frac{\sqrt{3}+1}{2} - \frac{\sqrt{3}-1}{2} = 1$ unit. This means that the combined area of the four sections is exactly 3 square units.

Q3

The square and the parallelogram are shown above.



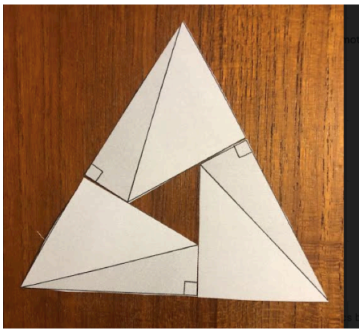


The total area of each shape must be 3 square units because there are no holes. The side of the large square must be $\frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2} = \sqrt{3}$ and the side of the square within it must be $\sqrt{2}$ and this implies that the total triangular area outside the inner square must be 1 square unit.

The parallelogram has area 3 square units and since its base length is 2 units its height must be $\frac{3}{2}$ units.

Q4

Here is the triangular shape



Each side has length $1 + \frac{\sqrt{3}+1}{2} = \frac{\sqrt{3}+3}{2}$ and thus its height by Pythagoras' theorem becomes $h = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}+3}{2}\right) = \frac{3(\sqrt{3}+1)}{4}$. If we include the hole in the middle, the area of the triangular shape becomes $\frac{1}{2} \times \frac{\sqrt{3}+3}{2} \times \frac{3(\sqrt{3}+1)}{4} = \frac{3(\sqrt{3}+3)(\sqrt{3}+1)}{16}$ or about 2.42404. Since the area of the three sections is $3 \times \frac{3}{4} = 2\frac{1}{4}$, then the hole must have an area of about 0.174 square units.

More information can be found about these tiles in Ian Stewart's captivating volume *Professor Stewart's Casebook of Mathematical mysteries*.

Puzzle 17: Two boys

Assuming having a boy is as likely as having a girl a common incorrect response to this question is $\frac{1}{2}$ because it seems that the sex of the first child should have nothing whatsoever to do with the sex of the second.

However if we knew nothing about Jane's children, other than the fact that she had two children, then we would need to consider four equally likely outcomes denoted by BB, BG, GB, GG. The rider 'one of them is a boy' excludes one of these possibilities and this means that there are three equally outcomes left. Thus the probability that the other child is a boy is in fact $\frac{1}{3}$.

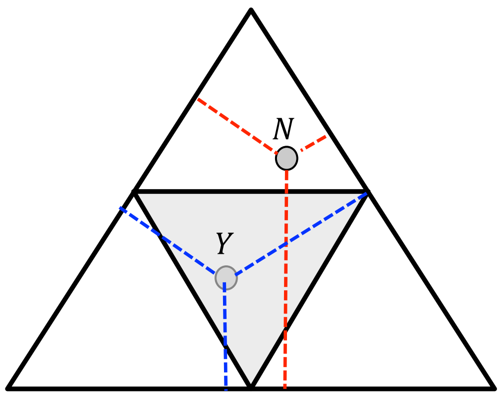
This argument might sound a little too theoretical at first, but if we actually contacted a random sample of four thousand two-children families, and then excluded any family with two girls, then we would find approximately one thousand of them having two boys.

Puzzle 18: From sticks to triangles

The triangle inequality theorem tells us that, to form a triangle, two of the pieces when put together end-to-end need to be longer than the third piece. So it's true that sometimes the broken pieces will not form a triangle.

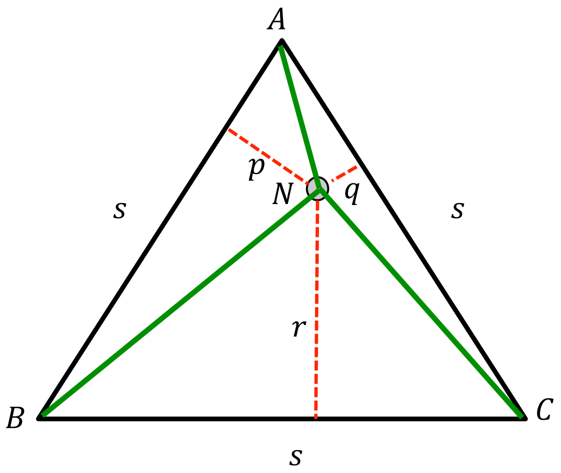
Worse still finding the probability that the three pieces form a triangle depends on how we break the pieces in the first place. One way to do this is to randomly mark two distinct points along the stick and then break the stick at those points. Another way is to randomly mark the stick at one point, break it there, and repeat the same procedure with the larger piece. We will discuss each of these methods separately.

The diagram below provides the key to unlocking the probability when two random points are chosen before any break is made. It shows an equilateral triangle with a smaller shaded inverted equilateral triangle inside it.



If we take any point inside the triangle (for example points N or Y shown in the figure) and draw perpendiculars to the three sides, the sum of these three sides will be constant and equal to the altitude of the triangle. This is a simple thing to prove using areas.

If we call the lengths of the three dashed lines from N as p, q and r and the length of the sides of the equilateral triangle s, then the sum of the areas of the three triangles ΔABN, ΔACN and ΔBCN is given by $\frac{1}{2}sp + \frac{1}{2}sq + \frac{1}{2}sr = \frac{1}{2}s(p + q + r)$. But the area of the entire equilateral triangle is $\frac{1}{2}sh$ where h is its altitude, and so by comparing these two equal values it immediately follows that $h = p + q + r$.



Looking back at N in the first figure it must be the case that $p + q$ is less than r because N is above the shaded triangle.

To make this clearer, if N was situated somewhere along the top line of the shaded triangle, then $p + q$ would equal the longest length r. Thus the top line of the shaded triangle acts as a sort of boundary where triangles can either be formed or not formed.

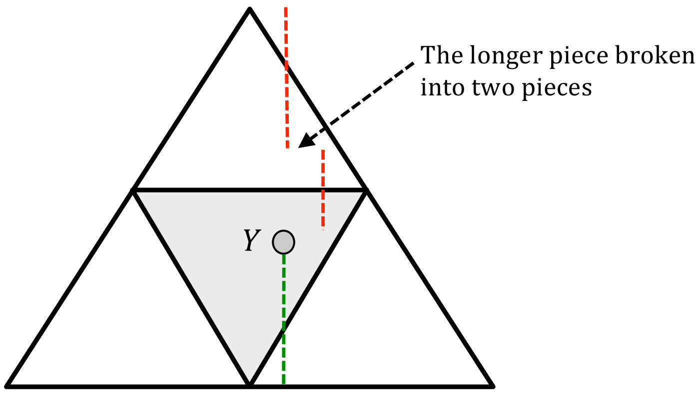
The same argument would apply analogously to all three sides of the shaded triangle. Hence the perpendiculars for any point inside the shaded triangle, say for example Y, will form a triangle.

In relation to our original question, the length of the original stick, say h, can be taken as the altitude of the triangle, so if you wish to experiment, simply put $s = \frac{2h}{\sqrt{3}}$ and draw the large equilateral triangle.

In other words the probability of the pieces forming a triangle is equal to the area of the shaded triangle divided by the area of the large equilateral triangle, which is plainly $\frac{1}{4}$.

Suppose instead that the pieces were formed by first breaking the stick at a random point along it, and then breaking the longer section at a random point on that piece (there is no point in breaking the shorter section because the two pieces formed are, when added together, immediately shorter than the longer piece and the pieces cannot therefore form a triangle).

Imagine that the intact shorter piece is the vertical green line in the diagram below. The other two red pieces, formed by breaking the longer piece are shown vertically to make it obvious that Y must be below the top line of the shaded triangle.



Therefore Y could range anywhere within the bottom three small triangles but the pieces will only form a triangle if Y is in the shaded triangle. Hence, given the longer piece is selected for breaking, the probability of the pieces forming a triangle increases to $\frac{1}{3}$. In any real situation, if the decision to break either the longer or shorter piece was a random one, then the probability of a triangle forming would become $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$. A very nice demonstration of the probabilities is shown here. Copy the address into your search engine.

https://twitter.com/MathsEdIdeas/status/979349140273532929

Puzzle 19: Sums of squares

This remarkable discovery was made by Charles Dodgson (aka Lewis Carroll) in October 1890 when he was mulling over the problem of finding two squares whose sum is also a square and chanced upon a theorem that (*his words*)

"if $x^2 + y^2$ be even, its half is the sum of two squares. A kindred theorem that $2(x^2 + y^2)$ is always the sum of two squares, also seems true and unprovable"

The first assertion states that if $x^2 + y^2$ is of the form $2n$ then it follows that there are whole numbers $u = \frac{1}{2}(x + y)$ and $v = \frac{1}{2}(x - y)$ such that $u^2 + v^2 = n$.

Thus $u^2 + v^2$

$$= \left[\frac{1}{2}(x + y)\right]^2 + \left[\frac{1}{2}(x - y)\right]^2$$

$$= \frac{x^2 + 2xy + y^2}{4} + \frac{x^2 - 2xy + y^2}{4} = \frac{1}{2}(x^2 + y^2) = n$$

For example, $5^2 + 7^2 = 74$ then it follows that $37 = 6^2 + 1^2$.

The kindred theorem that twice the sum of two squares can always be expressed as the sum of two squares is easy to prove, since if $u^2 + v^2 = \frac{1}{2}(x^2 + y^2)$ then doubling both sides shows that $2(u^2 + v^2) = x^2 + y^2$.

For example if $u = 3$ and $v = 4$ then $2(3^2 + 4^2) = 7^2 + 1^2$.

Puzzle 20: Gardner triples

Gardner triples $(a, b, c)_1$ consisting of three consecutive Fibonacci numbers (where a is any even positioned number in the Fibonacci sequence) are quite remarkable. The Fibonacci triples generalise the well-known *Three Square Puzzle* once made popular by Martin Gardner.

If $\tan^{-1}\frac{t}{a} = \tan^{-1}\frac{t}{b} + \tan^{-1}\frac{t}{c}$ then by taking tangents on both sides and using the appropriate compound angle formula we can show that $\frac{t}{a} = \frac{\frac{t}{b}+\frac{t}{c}}{1-\frac{t}{b}\times\frac{t}{c}} = \frac{t(b+c)}{bc-t^2}$ and thus $a = \frac{bc-t^2}{b+c}$ or when rearranged $c = \frac{ab+t^2}{b-a}$.

By defining a difference $d = b - a$ and substituting into this last expression we have $c = \frac{ab+t^2}{b-a} = \frac{a(a+d)+t^2}{d} = a + \frac{a^2+t^2}{d}$ and it becomes clear that a Gardner triple can be formed whenever d divides $a^2 + t^2$.
If we call $S = a^2 + t^2$ and re-label the difference d as d_1 so that $d_2 = \frac{S}{d_1}$ and choosing $d_1 < d_2$ then the triple becomes $(a, b = a + d_1, c = a + d_2)_t$.

For $a = 3$ and $t = 2$ we have $S = 13$ and the only divisors available are $d_1 = 1$ and $d_2 = 13$, so that the only type 2 Gardner triple becomes $(3, 4, 16)_2$. This means that $\alpha = \tan^{-1}\left(\frac{2}{3}\right) \approx 33.690°$, $\alpha = \tan^{-1}\left(\frac{1}{2}\right) \approx 26.565°$ and $\alpha = \tan^{-1}\left(\frac{1}{8}\right) \approx 7.125°$ and we see that 26.565° + 7.125° =33.690°.

The Fibonacci triples mentioned in the question are a subset of all possible Gardner triples. They are type $t = 1$ triples so that $c = \frac{ab+1}{b-a}$.
For example the triple $(3, 5, 8)_1$ shows $c = \frac{3\times5+1}{2} = 8$. The rest are verified in the same way.

By using an inductive proof it is possible to show that the complete set of of Fibonacci triples are given by $(F_{2n}, F_{2n+1}, F_{2n+2})_1$ where F_{2n} is any even-positioned Fibonacci number in the sequence.

Puzzle 21: Euler's boxes

Because there are six faces to every box the amount of cardboard used on the first five boxes must be $6\times(1)^2 + 6\times\left(\frac{1}{2}\right)^2 + 6\times\left(\frac{1}{3}\right)^2 + 6\times\left(\frac{1}{4}\right)^2 + 6\times\left(\frac{1}{5}\right)^2$ which simplifies to $6\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}\right] = 8.781\dot{6}\ m^2$. So nearly 88% of the 10 square metres Dr. Euler claimed have been used in the first five boxes. This seems extraordinary.

Dr. Euler's claim that that an infinite number of boxes will need an infinitely long space is true. The length required for the infinite line of boxes is given by the infinite harmonic series $S_1 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ an we can show that this series exceeds any number we care to name.

To see this, we could gather groups of terms together so that $S_1 = 1 + \frac{1}{2} + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) + \cdots$ and then replace some terms in the groupings with terms of a lower value, so that $S_1 > 1 + \frac{1}{2} + \left(\frac{1}{4}+\frac{1}{4}\right) + \left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right) + \cdots$ or when simplified $S_1 > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$. The right hand side of this last inequality could sum to any whole number we care to name and so S_1 will at some point become greater than that number. Hence it follows that number of boxes will require an infinite amount of space to line up in.

But what about the total amount of cardboard required?

In 1734 the great mathematician Leonard Euler proved that the infinite series given by $S_2 = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots$ converges to the sum $\frac{\pi^2}{6}$. What this means is that no matter how many terms of the series are included in any partial sum the total will never exceed $\frac{\pi^2}{6}$.

This can be a disconcerting notion to those readers who are not used to strange statements like this. After all, it seems a reasonable notion that adding positive quantities continuously, no matter how small they might be or get, should eventually become any total you care to name. However, this is not the case. For example if a frog jumped half way to a wall, and then jumped half the remaining distance to the wall again, and continued in this manner each time jumping half the remaining distance over and over again, it would be making forward progress forever, but never reach the wall. The wall acts as a limiting object that the frog continues to jump toward. Any point forward of the wall will be reached but the wall itself is an unbreachable fortress.

The total amount of cardboard used to make an infinite number of Dr. Euler's boxes is given by $6\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right) = 6\left(\frac{\pi^2}{6}\right) = \pi^2$ and the number $\pi^2 \approx 9.8696$. This explains his claim that the cardboard used will never exceed 10 square metres.