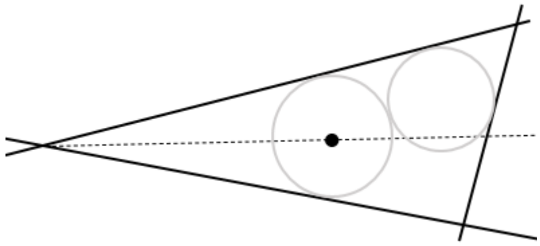
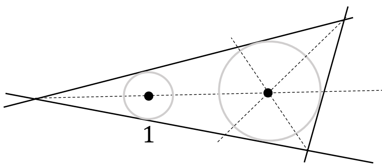
The six circles theorem

The Six Circles Theorem of C. Evelyn, G. Money-Coutts, and J. Tyrrell concerns a chain of circles inscribed in a triangle. A circle is inscribed in a vertex of a triangle. A second circle is drawn in either of the adjacent vertices so that it is tangent to the first.

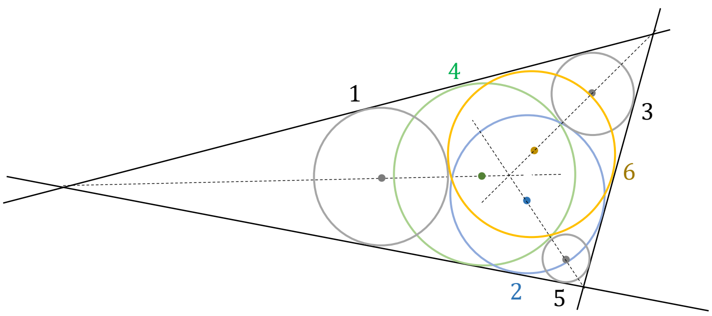


The process is continued. At each step, a new circle is placed in the next vertex so that it is tangent to the previous circle. It turns out that the sixth circle, which is tangent to the fifth, is also tangent to the first. Thus, the chain repeats after six steps.

The theorem in its basic form assumes that all the circles are inside the triangle and that the choice of the first circle is such that subsequent circles in the chain can exist. Note that the centres of the circles lie on the angle bisectors and that the incircle is the largest circle possible. Consequently, the first circle needs to be centred a distance from the incentre no more than twice the inradius. (The choice of the circle 1 in the following diagram would be impossible.)

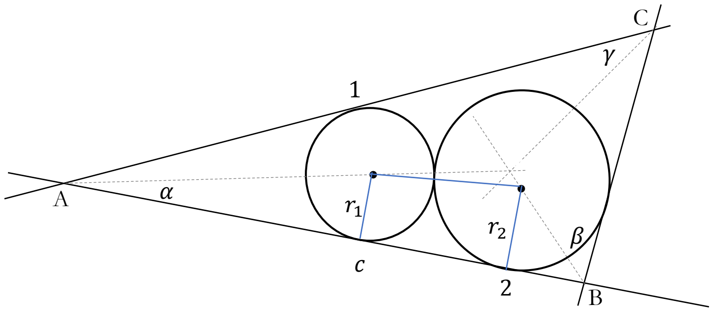


After a suitable choice for the initial circle and having decided whether to proceed clockwise or anticlockwise around the vertices, all subsequent circles exist and are uniquely determined.



A carefully drawn diagram appears to show that the claim is indeed a theorem, but a proof is needed.

Consider circles 1 and 2, inscribed in vertices A and B. The half-angles are respectively α and β, and the radii are r_1 and r_2. Side AB, tangent to the circles, has length c.



With the cotangent ratio and an application of Pythagoras' theorem, we deduce the relation

$$c = r_1 \cot\alpha + 2\sqrt{r_1 r_2} + r_2 \cot\beta \qquad (1)$$

This is symmetrical in the two radii, so that the positions of circles 1 and 2 can be exchanged without affecting the tangencies.

However, the third circle, in vertex C, must be different in each case. On one hand, we would have

$$a = r_2 \cot\beta + 2\sqrt{r_2 r_3} + r_3 \cot\gamma$$

and on the other,

$$a = r_1 \cot\beta + 2\sqrt{r_1 r_3'} + r_3' \cot\gamma$$

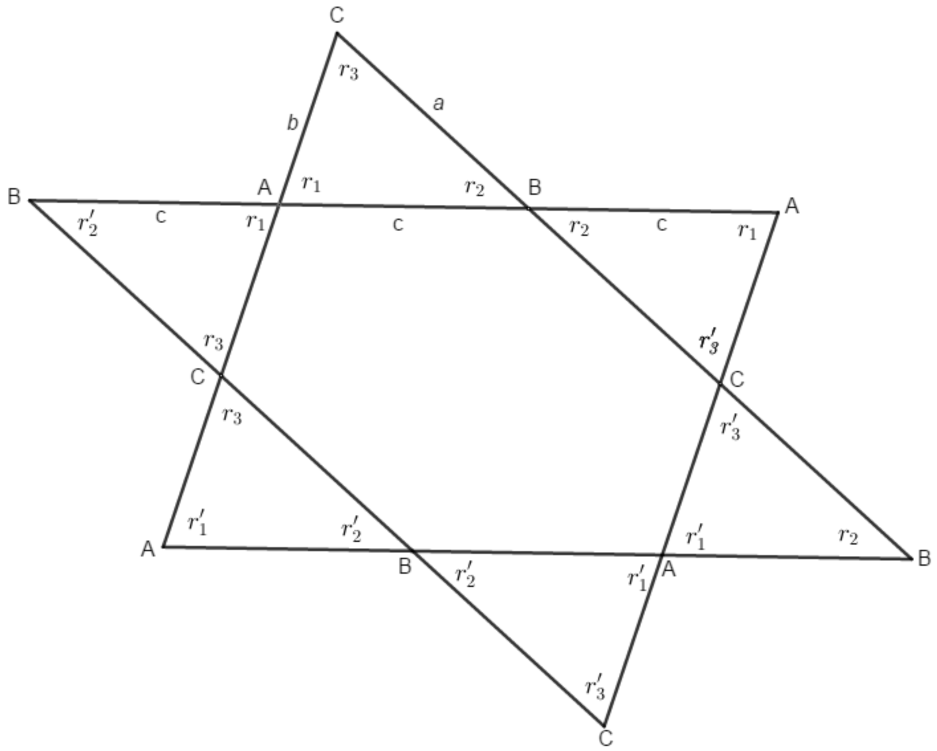
Clearly, if $r_2 > r_1$ it must be the case that $r_3' > r_3$ and vice versa.

Thus, in a triangle with vertices ABC, with inscribed circles of radii r_1, r_2, r_3 we obtain the triple r_2, r_1, r_3' by exchanging r_1 and r_2. Then, by exchanging r_2 and r_3' we obtain r_3', r_1', r_2. Continuing in this way, we generate further triples that satisfy the tangency constraints for the given triangle.

From eight possibilities, we select the six triples

$$r_1, r_2, r_3$$
$$r_2, r_1, r_3'$$
$$r_3', r_1', r_2$$
$$r_1', r_3', r_2'$$
$$r_2', r_1', r_3$$
$$r_3, r_1, r_2'$$

These can be arranged in a cyclic pattern. Rotations of copies of the original triangle through the angle π about the vertices are translated to make a chain joined by identical vertices, in the form of a six-pointed star, as in the following diagram.



To complete the proof of the six circles theorem, we need only disconnect this chain and superimpose the congruent triangles, rotating and translating as necessary so that their vertices coincide. The six circles within the triangles will then have the required tangency properties. That is, there are six circles such that each is tangent to two sides and to one other circle, and these form a repeating sequence of at most six distinct touching circles: $r_1, r_2, r_3, r_2', r_1', r_3', r_1$.

The book by Evelyn, Money-Coutts and Tyrell, cited below, in which the theorem was first published, has proved inaccessible to this author. It has therefore not been possible to determine to what extent the approach taken here is similar to that of the theorem's discoverers. The second reference, to the American Mathematical Monthly article, assumes some prior knowledge and demands a considerable effort from the reader. It follows a different tack from the one explored here towards its goal of demonstrating that a version of the theorem holds when the circles are not necessarily confined to the insides of the triangle.

Evelyn, C. J. A.; Money-Coutts, G. B.; Tyrrell, John Alfred (1974). The Seven Circles Theorem and Other New Theorems. London: Stacey International. pp. 49–58. ISBN 978-0-9503304-0-2.

Ivanov, Dennis; Tabachnikov, Serge (2016). "The six circles theorem revisited". American Mathematical Monthly. ***123*** *(7): 689–698. arXiv:1312.5260. doi:10.4169/amer.math.monthly.123.7.689. MR 3539854.*