The String Cut Problem: Circle and Square Solution (Paul Turner, November 2020)

A flexible string of length L is cut into two pieces. One piece is formed into a square and the other into a circle. Where should the string be cut to minimise the combined areas of the circle and square?

It is possible to form a circle with the full length of the string. Its radius is $\frac{L}{2\pi}$, and hence its area is $\frac{L^2}{4\pi}$. If a square is formed with the full string length, its area is $\frac{L^2}{16}$. As expected, the area of the square is less than that of the circle, and one might speculate that of all combinations of square and circle, the square by itself would have least area.

However, suppose the string is cut into equal parts of length $\frac{L}{2}$. Then the resulting circle has radius $\frac{L}{4\pi}$ and area $\frac{L^2}{16\pi}$, while the square has area $\frac{L^2}{64}$. The combination of the two areas is $\frac{L^2}{16} \cdot \frac{\pi+4}{4\pi}$ which is clearly smaller than the area of the square alone. Hence, there must be a minimum combined area reached when the string is cut somewhere between its endpoints.

Finding the minimum point is a standard classroom calculus exercise, but there is an insight that leads to the correct conclusion without the need for differentiation.

Let the two string-lengths be a and $L-a$, corresponding to the perimeters of the square and circle, respectively. If either is changed by a small amount δ, the other is changed by $-\delta$, and there are consequent small changes in the respective areas.

The critical observation is that the minimum combined area is reached when a small increase in one area is precisely matched by the resulting decrease in the other. If this were not so, it would be possible to attain a smaller total by moving the cut in one or the other direction.

For the square, we can write the area $\frac{a^2}{16}$ and for the circle, $\frac{(L-a)^2}{4\pi}$. Then, when $a \to a+\delta$, the area of the square changes by $\frac{(a+\delta)^2}{16} - \frac{a^2}{16} = \frac{2a\delta+\delta^2}{16}$ and the area of the circle changes by $\frac{(L-a-\delta)^2}{4\pi} - \frac{(L-a)^2}{4\pi} = \frac{-2(L-a)\delta+\delta^2}{4\pi}$. We require these changes to sum to zero as $\delta \to 0$.

Therefore, $\frac{2a\delta+\delta^2}{16} = \frac{2(L-a)\delta+\delta^2}{4\pi}$ and thus, in the limit,

$$\pi a = 4(L-a)$$

The length a is four times the side x of the square, and $L-a$ is 2π times the radius r of the circle. Hence, $4\pi x = 4 \cdot 2\pi r$ and we conclude that

$$x = 2r$$

That is, the circle is inscribed in the square.