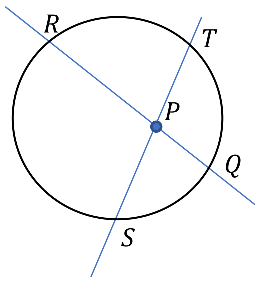
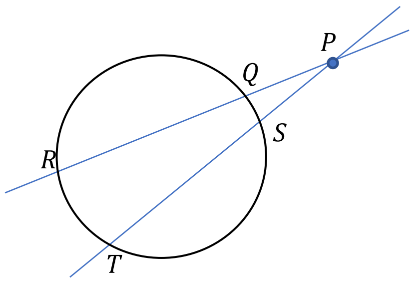
Take any circle and any point P, not on its circumference, lying in the same plane. Draw two concurrent lines through P that intersect the circle, the first at Q and R and the second at S and T as shown here as two cases.



Case 1



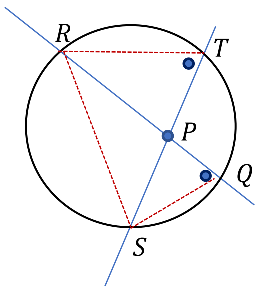
Case 2

Then in each case $PQ \times PR = PS \times PT$.

The theorem was first considered by the Swiss mathematician Jacob Steiner (1796 – 1863) whose mathematical work was mainly confined to geometry – considered by many to be the greatest pure geometer since Apollonius of Perga.

Here is an outline of the proof of case 1 above.

Draw in the line segments TR, QS and RS as shown. Then $\Delta RTP \cong \Delta SQP$, since $\angle RTP = \angle PQS$ (both standing on the chord RS) and $\angle RPT = \angle SPQ$. Thus, the ratios of corresponding sides are equal so that $\frac{PQ}{PT} = \frac{PS}{PR}$. The result immediately follows.



The theorem (both cases) is a powerful one because of its generality. Case 1, for example, can be used to explain the geometric mean. With QR a diameter and ST any chord at right angles to it, then $QP \times PR = PT \times PS = PT^2$ and so $PT = \sqrt{QP \times PR}$. Case 2, for example, explains that if PQ is a tangent (so that Q and R are coincident) then $PS \times PT = PQ^2$. If both PQ and PS are tangents, then $PQ = PS$. Other results follow as well.

The most difficult thing about a geometric proof is knowing where to start. Whilst many theorems involve probing for congruency and similarity, in the words of the American author Stephen Covey, it is always prudent to begin with the end in mind.

Challenge 10: Prove Case 2.