Charles Dodgson (1832-1898) better known by his pen name Lewis Carroll, devised a fascinating method for calculating determinants. Naming it the 'method of condensation', Dodgson acknowledged that he had got the idea from a theorem proved several years earlier by the German mathematician Carl Gustav Jacobi (1804 – 1851). While that theorem is not described here, we can demonstrate the method with an order-4 example. It has four main steps.

Consider the 4×4 matrix A given by

$$A = \begin{bmatrix} 2 & 1 & 1 & 4 \\ 1 & 2 & 1 & 6 \\ 1 & 1 & -2 & -4 \\ 2 & 1 & -3 & -8 \end{bmatrix}$$

The elements highlighted in red form the 'interior' matrix $\text{Int}(A) = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$. It plays a special role at the third step of the procedure. In general, $\text{Int}(M)$, for M an $n \times n$ matrix, will be an $(n-2) \times (n-2)$ matrix.

The **first step** is to form the matrix B (of an order *one less than* matrix A) made up of the $3^2 = 9$ 2×2 determinants of adjacent terms of A. We have maintained the colour coding to make it clear how each set of four elements are gathered.

$$B_{11} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \quad B_{12} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \quad B_{13} = \begin{vmatrix} 1 & 4 \\ 1 & 6 \end{vmatrix} = 2$$

$$B_{21} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1 \quad B_{22} = \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -5 \quad B_{23} = \begin{vmatrix} 1 & 6 \\ -2 & -4 \end{vmatrix} = 8 \qquad \text{Thus } B = \begin{bmatrix} 3 & -1 & 2 \\ -1 & -5 & 8 \\ -1 & -1 & 4 \end{bmatrix}$$

$$B_{21} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \quad B_{22} = \begin{vmatrix} 1 & -2 \\ 1 & -3 \end{vmatrix} = -1 \quad B_{23} = \begin{vmatrix} -2 & -4 \\ -3 & -8 \end{vmatrix} = 4$$

The **second step** is to repeat the same procedure with matrix B to produce a new matrix C, again one order less than B. Matrix B has one interior element, and so $\text{Int}(B) = (-5)$. There are $2^2 = 4$ adjacent determinants determined as

$$C_{11} = \begin{vmatrix} 3 & -1 \\ -1 & -5 \end{vmatrix} = -16 \quad C_{12} = \begin{vmatrix} -1 & 2 \\ -5 & 8 \end{vmatrix} = 2 \qquad \text{Thus } C = \begin{bmatrix} -16 & 2 \\ -4 & -12 \end{bmatrix}$$

$$C_{21} = \begin{vmatrix} -1 & -5 \\ -1 & -1 \end{vmatrix} = -4 \quad C_{22} = \begin{vmatrix} -5 & 8 \\ -1 & 4 \end{vmatrix} = -12$$

The **third step** is to divide each element of C by the corresponding elements of $\text{Int}(A)$ to form the $(n-2) \times (n-2)$ matrix C', and then evaluate its determinant.

$$|C'| = \begin{vmatrix} \frac{-16}{2} & \frac{2}{1} \\ \frac{-4}{1} & \frac{-12}{-2} \end{vmatrix} = \begin{vmatrix} -8 & 2 \\ -4 & 6 \end{vmatrix} = -40.$$

The **fourth step** is to divide $|C'|$ by $\text{Int}(B)$ so that $|A| = \left|\frac{-40}{-5}\right| = 8$.

Note that for a matrix M there may be instances where one or more elements of $\text{Int}(M)$ are zero, rendering a division impossible. This is fixed by altering M at the beginning with suitable elementary row operations that don't change $|M|$. Recall that if a multiple of any row is subtracted from another row, the value of the determinant is unchanged.

Challenge 12: Prove that the method is correct for all order 3 determinants.