There are quite a number of proofs of the **irrationality of $\sqrt{2}$** in the mathematical literature. All of them begin by assuming that the number is rational and then showing, through a series of logical steps, that the assumption leads to a contradiction.

For positive integers p and q, putting $\sqrt{2} = \frac{p}{q}$ leads easily to $p^2 = 2q^2$. From that point, the various methods take different and interesting pathways. One of those methods, for example, depends on the fundamental law of arithmetic. The argument is that because p^2 contains an even number of 2s in its unique prime factorisation, it cannot be equal to $2q^2$ because $2q^2$ contains an odd number of them.

Another develops a parity argument. Given $p^2 = 2q^2$ with $\frac{p}{q}$ in lowest terms, if q is even, then p is even, and there is an immediate contradiction because $\frac{p}{q}$ is already in lowest terms. If q is odd however, then p is still even, but then $p^2 = 2q^2$ becomes $(2m)^2 = 2(n+1)^2$ and that also leads to a contradiction.

Another type of proof is known as a **proof by infinite descent**. Here are two examples for the same positive integers p and q.

Proof 1

From $p^2 = 2q^2$ it is obvious that $p > q > 0$, and, because the right-hand side is clearly even, so also must p be even. So, for integer r, let $p = 2r$ so that $4r^2 = 2q^2$ simplifying to $q^2 = 2r^2$ (note that $p > q > r > 0$).

However, this last equation has exactly the same form as the first one. Therefore, repeating the procedure for another integer s leads to $r^2 = 2s^2$ with $p > q > r > s > 0$. Each time the new integer gets smaller and smaller so that eventually the stock of available integers is exhausted irrespective of the choice of our starting integer p. It is a descent into an inevitable contradiction.

Proof 2

From $p^2 = 2q^2$, subtract pq from both sides and factorise so that $p(p-q) = q(2q-p)$ and thus $\frac{p}{q} = \frac{2q-p}{p-q}$. Since we know that $1 < \sqrt{2}$ we can write that $1 < \frac{p}{q}$ and consequently that $q < p$, $2q < 2p$ and, subtracting p from both sides, $2q - p < p$. Also, since we know that $\sqrt{2} < 2$ we can write that $\frac{p}{q} < 2$ and consequently $p < 2q$ and thus $p - q < q$.

But this implies that there are two distinct but equal rational expressions for $\sqrt{2}$, with the new expression, say $\frac{p_2}{q_2} = \frac{2q-p}{p-q}$ having both p_2 and q_2 respectively less than the numerator and denominator of $\frac{p}{q}$.

We could repeat the same procedure for $\frac{p_2}{q_2}$ to derive a third expression $\frac{p_3}{q_3}$ for $\sqrt{2}$ whose numerator and denominator would each be less than those of $\frac{p_2}{q_2}$. This process will descend to a contradiction.

Challenge 13: Prove the irrationality of $\sqrt{p}$ for all prime p by using a proof that depends on the fundamental theorem of arithmetic.