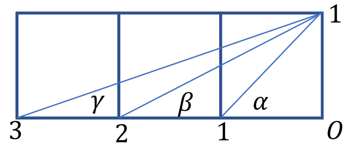
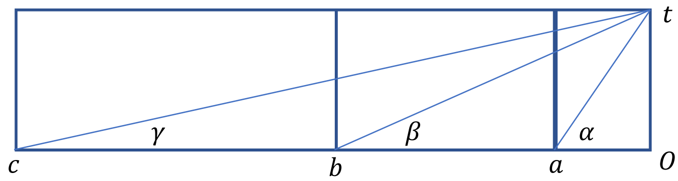
The celebrated three-square problem, shown below, involves proving geometrically, for angles α, β and γ, that $\alpha = \beta + \gamma$ as shown here.



In a series of published articles (Australian Senior Mathematics Journal, 2019), Paul Turner and I generalised this problem by considering, instead of unit squares, rectangles of varying width and constant height. Specifically, for positive integers a, b, c and t we sought solutions to the equation $\tan^{-1}\left(\frac{t}{a}\right) = \tan^{-1}\left(\frac{t}{b}\right) + \tan^{-1}\left(\frac{t}{c}\right)$ where $\alpha = \tan^{-1}\left(\frac{t}{a}\right)$, $\beta = \tan^{-1}\left(\frac{t}{b}\right)$, and $\gamma = \tan^{-1}\left(\frac{t}{c}\right)$. The diagram depicts the generalisation.



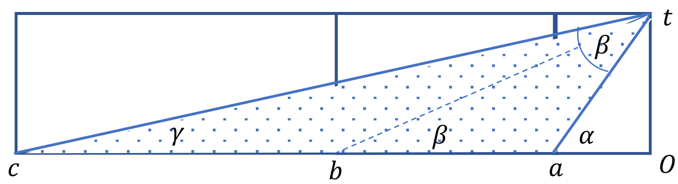
A convenient notation for solutions was adopted and named *Gardner Triples*, after Martin Gardner, the famous American recreational mathematics writer. Any type t solution was referred to as $(a, b, c)_t$ with a simpler notation (a, b, c) when $t = 1$. For example, the original three-square problem would be written $(1,2,3)$. The method of obtaining solutions is described as follows.

Choose two coprime integers a and t. Sum their squares to obtain S. Find integers m and n, $m < n$, such that $mn = S$. Then $(a, a+m, a+n)_t$ is a type t Gardner triple. (See Tip 15 for an explanation.)

For example, setting $a = 1$ and $t = 1$, we have $S = 2$ and thus, $(m, n) = (1,2)$ and the type 1 Gardner triple is $(1,2,3)$, as depicted in the original three-square puzzle. As another example, choosing $a = 3$ and $t = 5$ shows $S = 34$ so there are two candidate factor pairs (m, n), namely $(1,34)$ and $(2,17)$. These lead to solutions of $(3, 4, 37)_5$ and $(3, 5, 20)_5$. In terms of the angles, this means that $\tan^{-1}\left(\frac{5}{3}\right) = \tan^{-1}\left(\frac{5}{4}\right) + \tan^{-1}\left(\frac{5}{37}\right)$ and $\tan^{-1}\left(\frac{5}{3}\right) = \tan^{-1}(1) + \tan^{-1}\left(\frac{1}{4}\right)$.

Every set of three consecutive Fibonacci numbers $F_{2n}, F_{2n+1}, F_{2n+2}$ is a type 1 Gardner triple. For example, the original problem $(1,2,3)$ starting from F_2 is a Gardner triple. Likewise, $(3,5,8)$, $(8,13,21)$, etc. are all Gardner triples, but there are many more families of triples to discover.

As a final note, the authors defined a Gardner Triangle as a triangle with internal angles having rational tangents, as shown shaded below. Note that the external angle $\alpha = \beta + \gamma$ is the sum of the two interior opposite angles and thus the tangents of the internal angles are all rational.



Challenge 14: Find a Gardner triple using $a = 5$ and $t = 2$.