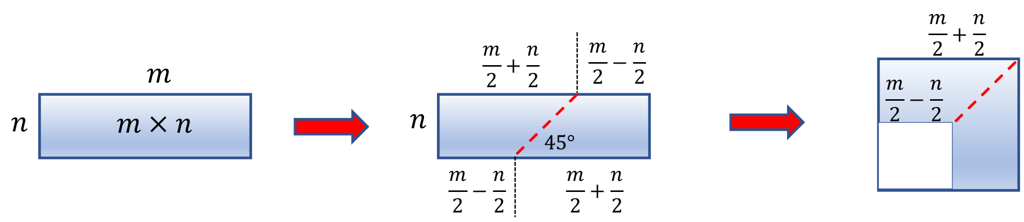
TIP 20: The Babylonian Quadratic Formula

The quadratic formula $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ is standard fare for most high school mathematics courses. It is the solution to the quadratic equation $ax^2 + bx + c = 0$.

An authentic way to introduce the formula originates from an ancient Babylonian discovery that every product of two positive integers can be expressed as the difference of two squares. In modern terms, the product mn is found by calculating two quantities, $\frac{m+n}{2}$, and $\frac{m-n}{2}$. The product is then the difference between the squares of those two quantities.

For example, to calculate the product of the numbers 12 and 8, the half-sum is 10 and the half-difference is 2 and so the product is $10^2 - 2^2 = 96$. If the numbers were 12 and 9, the calculation becomes $10.5^2 - 1.5^2 = 108$. By using this technique, the multiplication of any two whole numbers requires a simple table of consecutive integer and half integer squares. The efficiency is realised when one considers that, using the Babylonian squares method, covering all possible integer products up to 100×100 requires a mere 400 entries whereas an $m \times n$ array requires at least $4{,}550$ entries.

In the diagram the product mn is first represented by the rectangle on the left. It is then cut in half along the slanted red line, shown in the middle diagram, and the two pieces are lifted, turned and put together into the L shaped position shown on the right.



As an algebraic statement we write $mn = \left(\frac{m}{2}+\frac{n}{2}\right)^2 - \left(\frac{m}{2}-\frac{n}{2}\right)^2 = \frac{(m+n)^2}{4} - \frac{(m-n)^2}{4}$

The technique can be used to solve the quadratic equation $x^2 + bx + c = 0$. To see this first write the equation as $(x + b)x = -c$ so that the left hand side is the product of the two numbers $(x + b)$ and x (We assume, based on the consideration of areas, that $x > 0$ and that $c < 0$. In modern interpretations these restrictions are unnecessary).

Thus we have $\frac{\left((x+b)+x\right)^2}{4} - \frac{\left((x+b)-x\right)^2}{4} = -c$ or, when simplified, $\frac{(2x+b)^2}{4} - \frac{b^2}{4} = -c$. Hence $\frac{(2x+b)^2}{4} = \frac{b^2}{4} - c = \frac{b^2-4c}{4}$ so that $2x + b = \pm\sqrt{b^2 - 4c}$ and finally $x = \frac{-b\pm\sqrt{b^2-4c}}{2}$.

Note that the more general equation $ax^2 + bx + c = 0$ is equivalent to $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, so that by replacing b with $\frac{b}{a}$ and c with $\frac{c}{a}$ the solution becomes $x = \frac{-\frac{b}{a}+\sqrt{\frac{b^2}{a^2}-4\left(\frac{c}{a}\right)}}{2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Challenge 20: Research Quarter Square Tables. How were these tables used in the industrial age?