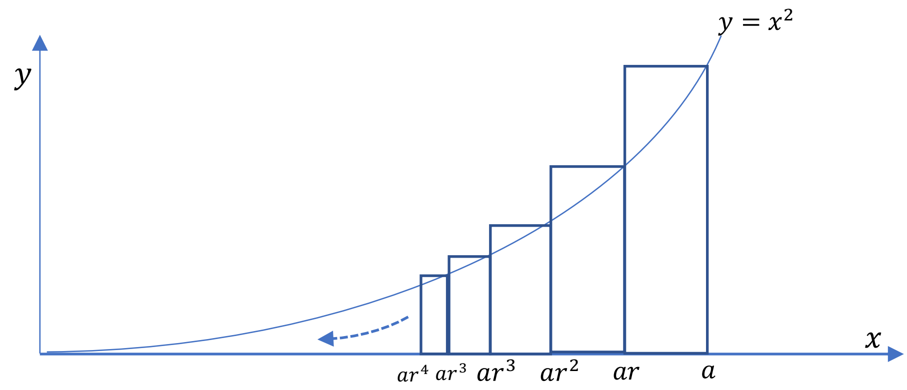
In 1666, Isaac Newton (1643-1727) calculated areas by antidifferentiation and this work contains the first clear statement of the Fundamental Theorem of the Calculus. Perhaps less well known is a result made by Pierre de Fermat (1601-1665) as early as 1629 that uses geometric sums in a novel way to find the area under the curve $y = x^2$. The solution would make an ideal introduction to the first steps in integral calculus.

It starts with a graph of the function between $x = 0$ and $x = a$ and a few of the circumscribed rectangles, but in Fermat's solution the width of each rectangle, constructed from the right, diminishes so that the left hand edges of each rectangle line up with x values that form a geometric progression whose first term is a and common ratio is r where $r < 1$.



Even though the rectangles continue indefinitely toward the y axis we can sum their areas for we know their widths, given as $(a - ar), (ar - ar^2), (ar^2 - ar^3)$, ... and their heights, given as $f(a) = a^2, f(ar) = a^2r^2, f(ar^2) = a^2r^4$, ...

Thus, after taking a^3 out as a common factor, the sum S can be written:

$$S = a^3[(1 - r)(1 + r^3 + r^6 + \cdots)]$$

The summation in the second bracket is infinitely geometric with first term 1 and common ratio $r^3 < 1$ and so has the limiting sum $\frac{1}{1-r^3} = \frac{1}{(1-r)(r^2+r+1)}$. When this expression is substituted in, the sum becomes

$$S = \frac{a^3}{(r^2 + r + 1)}$$

At this point, Fermat's genius kicks in. If r is allowed to approach 1 the widths of all of the rectangles reduce, and the parts of the rectangles above the curve reduced also. It is not difficult to imagine that the sum of the areas of all of the rectangles approaches the area under the curve.

This implies that $r^2 + r + 1 \to 3$ and $S \to \frac{a^3}{3}$.

Challenge 3: Show equivalently, for the function $y = x$, $S \to \frac{a^2}{2}$.