High school students learn to sketch linear functions of the form $y_1 = mx + b$ by tabulating, plotting (perhaps three or five coordinate points) and then drawing the imagined line. The y values of each point are found by substituting the respective x values into the function and performing the necessary calculations. This two-step process involves one multiplication and one addition for each of the points.

The task becomes harder when plotting points along the quadratic curve $y_2 = ax^2 + bx + c$. Evaluating the y values directly becomes a five-step process with three multiplications and two additions. Likewise, finding y values for the cubic function $y_3 = ax^3 + bx^2 + cx + d$ takes nine steps. For an n degree polynomial we can show that at most $\frac{n(n+3)}{2}$ steps are required to evaluate y values directly, so that the work involved increases according to the square of the degree.

In days when such calculations were performed manually, sketching higher order polynomials by plotting evaluated function values would have been both time consuming and subject to the risk of calculation error. George William Horner (1786-1837), English headmaster and mathematician, found a way to substantially reduce the number of calculations involved.

He noticed that $y_2 = ax^2 + bx + c$ could be re-expressed as $y_2 = (ax + b)x + c = Px + c$ where evaluating $P = ax + b$ involved 2 steps and therefore the overall calculation involved only 4 steps rather than 5. Similarly, the cubic function $y_3 = ax^3 + bx^2 + cx + d$ becomes $y_3 = Px^2 + cx + d = (Px + c)x + d = Qx + d$, where evaluating P and Q each involve 2 steps so that overall there are 6 steps rather than 9 in the calculation. Each new linear expression formed after the first uses the results obtained from the previous linear expression to create a chain of calculations that progressively build up to the final sum.

For example, to evaluate $y_5 = 3x^5 - 2x^4 + x^3 - 4x^2 + 7x - 5$ at, say $x = 2$, we progressively calculate

$$\begin{aligned} P &= 3(2) - 2 = 4 \\ Q &= 4(2) + 1 = 9 \\ R &= 9(2) - 4 = 14 \\ S &= 14(2) + 7 = 35 \\ T &= 35(2) - 5 = 65. \end{aligned}$$

That's ten calculations in all whereas a direct attack would have doubled the work. The progressive nature of Horner's scheme (as it came to be known) avoids the calculation of powers of numbers. Each step simply involves the sum of a coefficient and a multiple of the x value. The advantages are obvious.

Challenge 5: Prove that the number of steps required to evaluate the general degree n polynomial without using Horner's scheme is $\frac{n(n+3)}{2}$.