No multiple of an irrational number can be an integer, but can we find a multiple that is arbitrarily close to one? The question occurs in the topic Diophantine Approximation. Peter Gustav Lejeune Dirichlet applied his 'box' principle to it (often called the 'pigeonhole principle').

The pigeonhole principle allows us to conclude, for example, that we need only pull four socks from a drawer containing many socks of just three colours, to be sure of obtaining two of the same colour.

A more formal version of the approximation problem says:

> Let x be an irrational number lying between 0 and 1 and let Q be an arbitrarily large positive integer. Show that there is at least one pair of relatively prime positive integers p and q such that q is at most Q and $|qx - p| < 1/q$.

We use the notation $\{nx\}$ to mean a multiple nx with its integer part removed.

The interval $(0,1)$ is a union of Q subintervals, $\left(0, \frac{1}{Q}\right] \cup \left[\frac{1}{Q}, \frac{2}{Q}\right] \cup \ldots \cup \left[\frac{Q-1}{Q}, 1\right)$. Each subinterval has length $\frac{1}{Q}$. Consider the points (with integer parts removed): $\{x\}, \{2x\}, \{3x\}, \ldots, \{Qx\}, \{(Q+1)x\}$. We have chosen to consider $Q + 1$ of them but there are Q subintervals. Hence, there must be at least two multiples lying in the same 'box'. Thus, there are two multiples for which we have $\left|\{q_i x\} - \{q_j x\}\right| < \frac{1}{Q}$.

That is, there exist integers p_i and p_j such that

$$\left|(q_i x - p_i) - (q_j x - p_j)\right| = \left|(q_i - q_j)x - (p_i - p_j)\right| < \frac{1}{Q}$$

Assuming $q_i > q_j$, the quantity $q_i - q_j = q$ is at most Q, and $p_i - p_j = p$ is a positive integer. Therefore, $|qx - p| < \frac{1}{Q} \leq \frac{1}{q}$, as required. In fact, the inequality must hold for every pair q_i, q_j that differ by q, and so, many such pairs will share a box.

It follows that $\left|x - \frac{p}{q}\right| < \frac{1}{Qq} \leq \frac{1}{q^2}$.

Example

Let $x = \{\sqrt{3}\}$ and take $Q = 100$. With a spreadsheet it is easy to tabulate the products $\{x\}, \{2x\}, \ldots, \{101x\}$ and to sort them into ascending order. Products that are the same in the second decimal place fall into the same box. It turns out that $q = 56$ and $p = 41$.

Thus, $\frac{41}{56} \approx 0.73214 \ldots$ should be close to $\sqrt{3} - 1 \approx 0.73205 \ldots$.

The difference is not only less than $\frac{1}{56^2}$ but also less than $\frac{1}{101^2}$. If we truncate the decimal approximation of $\sqrt{3} - 1$ to two places we have $\frac{73}{100}$, but the fraction $\frac{41}{56}$ is closer to the true value and is a better approximation in the sense that it has a smaller denominator.

Challenge 6: Find a good rational approximation to $\frac{1}{\sqrt{2}}$ with a denominator less than 100.