One of the first modern *Tables of Quarter-Squares* was compiled by the Parisian auctioneer **Antoine Voisin** of whom very little else is known. They were in circulation in Paris around the year 1817. He called his quarter-squares “logarithms” although they were nothing like them. We can however think of them as number ‘codes’ that can be used to reduce the manual work of multiplication.

A table of quarter-squares was a set of integers and their codes. The user performed one addition and one subtraction, looked up two codes based on the results and subtracted them. Miraculously the product appeared. The idea caught on quickly - they were in wide-spread use in the 19th and early 20th century. The concept is illustrated in this short table (mn, $m + n \leq 50$ and $m \geq n$).

N	code	N	code	N	code	N	code	N	code
1	0	**11**	30	**21**	110	**31**	240	**41**	420
2	1	**12**	36	**22**	121	**32**	256	**42**	441
3	2	**13**	42	**23**	132	**33**	272	**43**	462
4	4	**14**	49	**24**	144	**34**	289	**44**	484
5	6	**15**	56	**25**	156	**35**	306	**45**	506
6	9	**16**	64	**26**	169	**36**	324	**46**	529
7	12	**17**	72	**27**	182	**37**	342	**47**	552
8	16	**18**	81	**28**	196	**38**	361	**48**	576
9	20	**19**	90	**29**	210	**39**	380	**49**	600
10	25	**20**	100	**30**	225	**40**	400	**50**	625

As an example, to multiply 12 by 7, the user first determined $12 + 7 = 19$ and $12 - 7 = 5$ and then looked up the codes for those results. The codes are 90 and 6 and their difference is 84, the required product. A square number like 12^2 would simply be the code for 24, listed as 144 assuming that the code for the zero difference was also 0. In this very small table of 50 codes, there are 1,225 products that can be determined. A typical 19th century table however might contain as many as 20,000 codes.

The codes for the product mn are simply the integer parts of the quarter-squares $\frac{1}{4}(m+n)^2$ and $\frac{1}{4}(m-n)^2$. Subtracting these codes as $\frac{m^2+2mn+n^2}{4} - \frac{m^2-2mn+n^2}{4}$ reveals their product mn.

Note that, depending on parities of m and n, the quarter-squares may contain a decimal part, but because these will cancel on the final subtraction, they are not required in the table.

That is to say, if m and n are both of the same parity (both even or both odd), their sum and difference will be even and so the quarter square code will not have a fractional part. If they are of different parity (one even and one odd), their sum and difference will have the same fractional part and thus those parts will cancel when the final subtraction happens. It is an ingenious solution for an age that didn’t have access to electronic calculators and computers.

An 1889 article written in *Nature* magazine (issues 10 and 17 October) reads:

A multiplication of ‘double entry’ reaches only to 1000 by 1000 and forms a closely printed folio of 900 pages, but a table of quarter squares of the same extent need only occupy four octavo pages.

Challenge 7: How many products are possible in any list of n integers and codes (ignoring commutativity)