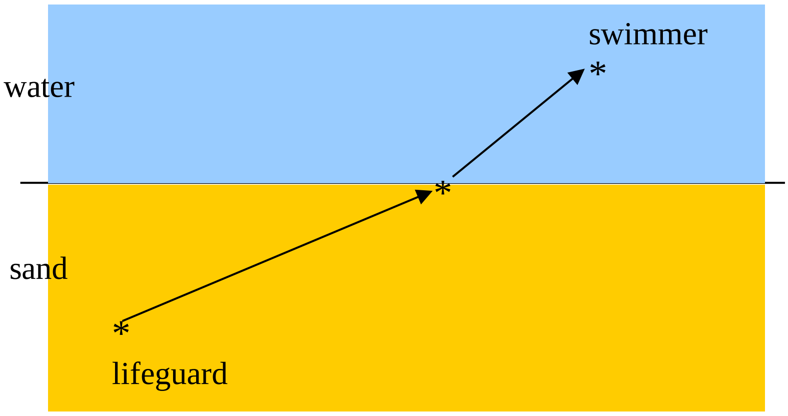
The beach and the brachistochrone

Introduction

In explanations for the law of refraction of light, the analogy has been used of a lifeguard on a beach who must get to a swimmer in trouble as quickly as possible. The lifeguard must run across the sand to the water and then swim. The speeds of the lifeguard across sand and water are different so that to minimise the time taken to get to the swimmer the lifeguard should choose a path that is not necessarily direct but may be kinked at the edge of the water.



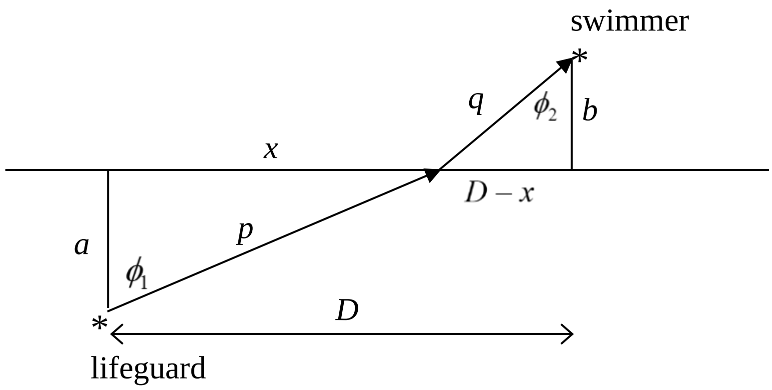
In the case of a ray of light, the phenomenon of refraction and its physical law was known to the mathematician and physicist Ibn Sahl [1] living in Baghdad around 984. Over five centuries later, the law was rediscovered by Thomas Harriot (1602), Willibrord Snellius (1621), Rene Descartes (1637) and Pierre de Fermat (1638).

Fermat derived the refraction law based on his principle of least action – that the natural path of a light ray would minimise the time taken – without an explanation as to why this should be so. For the derivation he used a procedure he called *adequality* [2], an idea not far removed from the differential calculus, that enabled him to find maxima or minima and tangents to curves. Thus, Fermat showed that the observed refraction law was consistent with the least-time principle. A plausible physical explanation of this puzzling result was provided years later, in terms of wave mechanics, by Christiaan Huygens (1678) [3], and much later again in terms of modern quantum electrodynamic theory.

In this article we derive Snell's law using the modern form of differential calculus with the lifeguard example as a model. We go on to consider optimum paths when the softness of the sand varies, that is, the medium is anisotropic. Finally, we use the same idea and a procedure somewhat like one used by Johann Bernoulli (1697) to obtain the mathematical form of the brachistochrone.

The lifeguard

Let the lifeguard's speed across sand be v_1 and across water be v_2. Using the symbols in the following diagram, an expression can be written for the total time of the lifeguard's excursion in terms of the variable distance *x*.



Thus, $p=\sqrt{a^2+x^2}$ and $q=\sqrt{b^2+(D-x)^2}$.

The time to cover distance *p* over sand is

$$t_1=\frac{\sqrt{a^2+x^2}}{v_1}$$

and the time to cover distance *q* through water is

$$t_2=\frac{\sqrt{b^2+(D-x)^2}}{v_2}$$

Hence, the total time taken, T, is given by $T=t_1+t_2$. We need the value of *x* that minimises *T*. Differentiating *T* with respect to *x*, we obtain

$$\frac{dT}{dx}=\frac{x}{v_1\sqrt{a^2+x^2}}-\frac{D-x}{v_2\sqrt{b^2+(D-x)^2}}=\frac{x}{v_1p}-\frac{D-x}{v_2q}=\frac{\sin\phi_1}{v_1}-\frac{\sin\phi_2}{v_2}$$

Hence, on equating the derivative to zero, we have $\frac{\sin\phi_1}{\sin\phi_2} = \frac{v_1}{v_2}$ and if x is chosen so that this is satisfied, the time T is minimised. The relation does not immediately provide the least-time path but it is of interest since it has the form of Snell's law of refraction of light, which is observed experimentally.

Snell's law

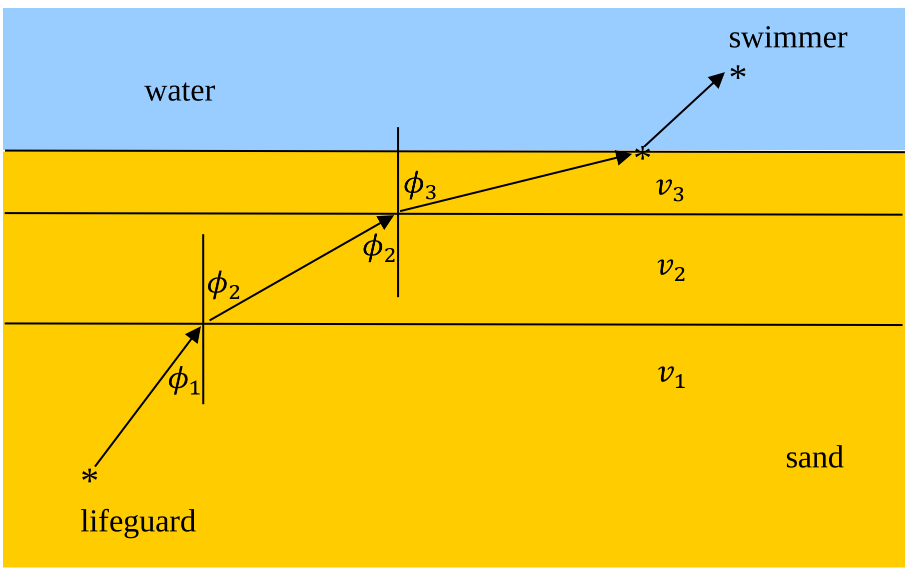
Snell's law is usually stated in terms of refractive indices, $n = \frac{c}{v}$, where c and v are respectively the velocities of light in empty space and in the medium in question. Thus, $v = \frac{c}{n}$ and the derivation above is equivalent to $\frac{\sin\phi_1}{\sin\phi_2} = \frac{n_2}{n_1}$.

Since this condition holds, it appears, by analogy with the lifeguard, that a ray of light travelling from one medium to another takes a least-time path.

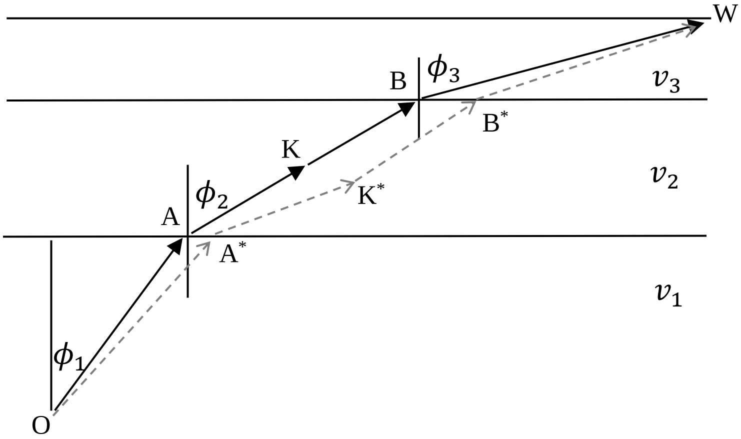
Varying speeds over the sand

Complicating the situation, we might imagine that the sand on the beach is not uniformly packed, so that the running lifeguard travels at different speeds according to the softness of the sand along the way.

Suppose there are, say, three regions of sand between the lifeguard station and the water such that the speed of running increases as the lifeguard crosses from one region to the next. The best path through the sand could be shaped approximately as in the following diagram.



We can show that the path OABW in the next diagram (extracted from the one above) with the angles determined at each of the boundaries according to Snell's law, is a minimum time path.



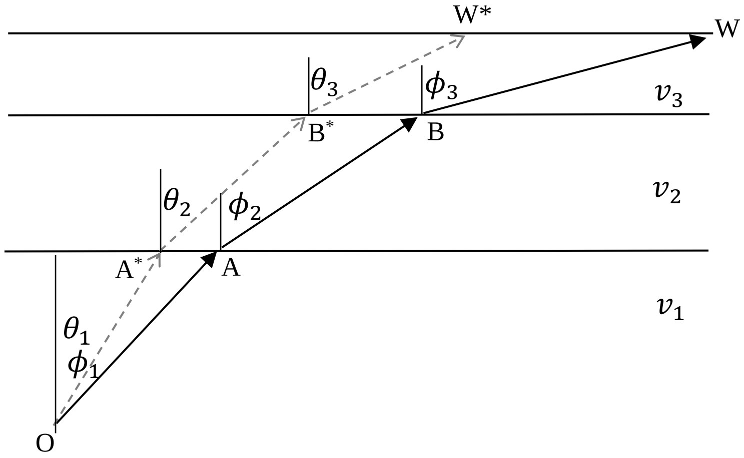
Suppose the time-minimising path for the lifesaver passes through K*, a point in the second of the three regions. In this region the speed of the lifesaver is v_2. Then, points A* and B* on the boundaries between the sand regions are determined by Snell's law to minimise the time for the paths OA*K* and K*B*W.

However, if A*, K* and B* are not collinear, there will be a faster, direct path between A* and B*. Hence, the time-minimising path for the lifesaver does not include K* in this case. Therefore, we must require K* to lie on a line segment AB that connects points on the two boundaries. For example, K* could coincide with K in the diagram.

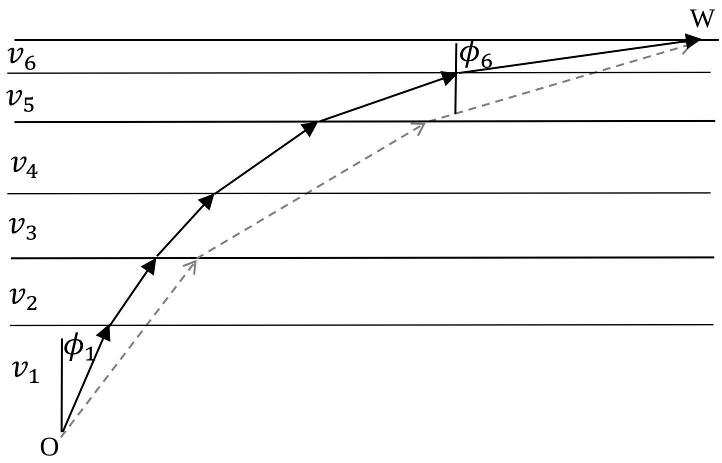
Points A and B are determined such that Snell's law is simultaneously satisfied at both boundaries and the path from A to B is a line. Thus, $\frac{\sin\phi_1}{\sin\phi_2} = \frac{v_1}{v_2}$ and $\frac{\sin\phi_2}{\sin\phi_3} = \frac{v_2}{v_3}$, as required. It follows, incidentally, by multiplying the two ratios, that

$$\frac{\sin\phi_1}{\sin\phi_3} = \frac{v_1}{v_3} \qquad (1)$$

Similarly, the diagram below shows how the lifeguard should approach the endpoint W. Suppose the minimum time path contains a point W* to the left of W at the edge of the water. Points A* and B* have been determined so that Snell's law is satisfied. Clearly, a direct path B*W is faster than B*W*W. Therefore, point W* cannot belong to the minimum time path unless it coincides with the terminal point W.



Similar arguments can be applied when more regions are introduced, each permitting a different running speed. The paths connecting successive boundaries between regions must be line segments and the optimal path must be a polygon with Snell's law determining the angles at each boundary.



As each of the speed regions is subdivided, with $v_1 < v_2 < \cdots < v_6$, a new optimal polygonal path appears. The process of subdividing the speed regions can be repeated until, in the limit, the speed of the lifeguard running through the sand varies continuously. The polygonal least-time path becomes a smooth curve.

Equation (1) holds throughout the limiting process and so, for two distinct points A and X on a smooth least-time path, but possibly not at the endpoints, we have

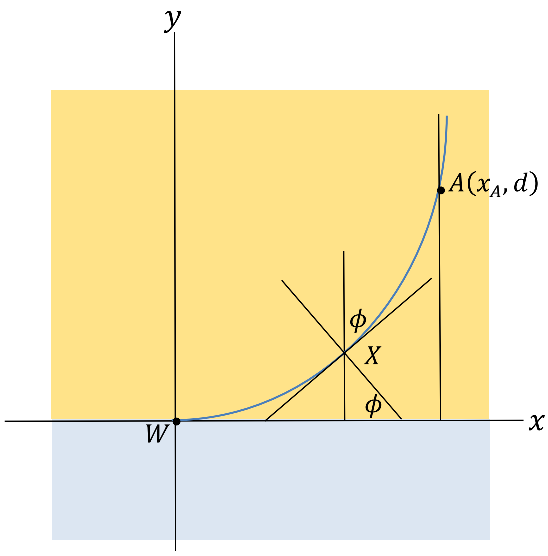
$$\frac{\sin \phi_A}{\sin \phi_X} = \frac{v_A}{v_X}$$

Taking A to be a fixed point makes $\frac{v_A}{\sin\phi_A} = \rho$ a constant, so that by (1) $\sin\phi_X$ is related to v_X by

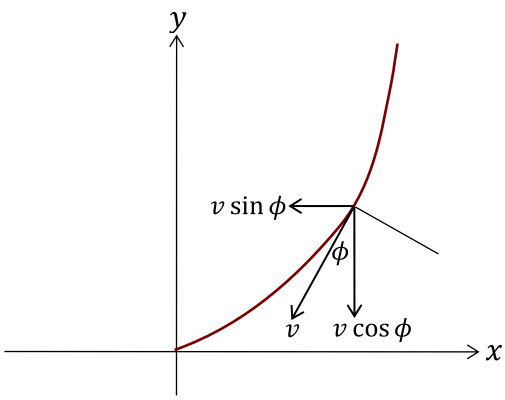
$$v_X = \rho \ \sin\phi_X \qquad (2)$$

where $0 \le \phi_X \le \frac{\pi}{2}$ is assumed.

For convenience, in what follows the image of the lifeguard's path across the sand has been re-oriented and we have inserted a coordinate system so that the endpoint W is at the origin. The lifeguard station is located at point $A(x_A, d)$.



We assume $v_{\min} > 0$ so that $\phi_X > 0$ everywhere on the path. At $v_{\max}$ it can happen that $\phi_X = \frac{\pi}{2}$ and from (2) it is clear that in this case $\rho = v_{\max}$. Otherwise, $\rho > v_{\max}$.



The scalar quantity v is the magnitude of the velocity vector tangential to the path at a time t, so that in a small interval δt we have $\delta x = v \sin\phi\, \delta t$ and orthogonal to it, $\delta y = v\ \cos\phi\, \delta t$.

Using (2) we have $\delta y = \rho \sin\phi \cos\phi\, \delta t$ and $\delta x = \ \rho \sin^2\phi\, \delta t$. Thus, in the limit, we have the equation $\frac{dy}{dx} = \frac{\cos\phi}{\sin\phi}$ at an arbitrary point on the path. Then,

$$\frac{dy}{dx} = \frac{\sqrt{1 - \left(\frac{v}{\rho}\right)^2}}{\frac{v}{\rho}} = \frac{\sqrt{\rho^2 - v^2}}{v} \qquad (3)$$

A particular speed function

The precise shape of the best path depends on how the runner's speed through the sand varies. In a somewhat unrealistic model, the speed v is related to $d - y$, the runner's distance from the lifeguard station in the direction directly towards the water, by a linear function in the form $v = a(d - y) + b$ where y varies from 0 to y_W.

We make the substitution $v = a(d - y) + b$ in equation (3). Thus, $\frac{dy}{dx} = \frac{\sqrt{\rho^2 - (a(d-y)+b)^2}}{a(d-y)+b}$, and this differential equation has the solution

$$x = \frac{1}{a}\sqrt{\rho^2 - (a(d - y) + b)^2} + C.$$

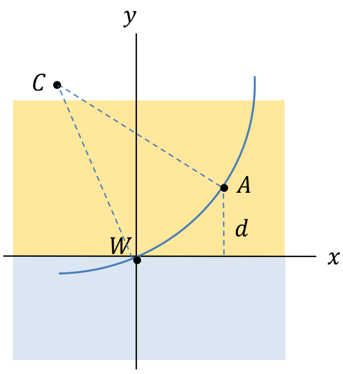
Since $y = 0$ when $x = 0$, we find $C = -\frac{1}{a}\sqrt{\rho^2 - (ad + b)^2}$. Then, after some manipulation,

$$(x - C)^2 + \left(y - \left(d + \frac{b}{a}\right)\right)^2 = \left(\frac{\rho}{a}\right)^2.$$

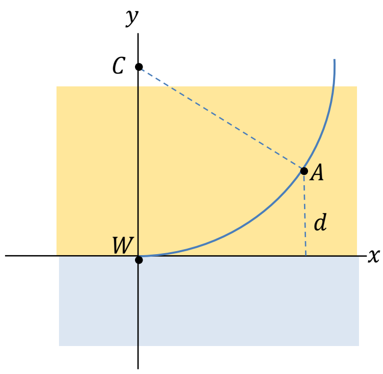
Hence, the least-time path is an arc of a circle with radius $r = \frac{\rho}{a}$ and centre $\left(-\frac{1}{a}\sqrt{\rho^2 - (ad + b)^2}, d + \frac{b}{a}\right)$.

Two ways in which a particular path could lie on this circular arc are illustrated below.

Case 1: $\rho > v_{\max}$



Case 2: $\rho = v_{\max}$



The values for a, b and $\rho = ar$ are to be determined.

When $y = d$, the condition $v = a(d - y) + b$ implies, $v_{\min} = b$.

Similarly, when $y = 0$, we have $ad + b = v_{\max}$. Therefore , $a = \frac{v_{\max} - v_{\min}}{d}$

In each case, the path belongs to the circle

$$\left(x + \frac{1}{a}\sqrt{\rho^2 - (ad + b)^2}\right)^2 + \left(y - \left(\frac{ad + b}{a}\right)\right)^2 = \left(\frac{\rho}{a}\right)^2.$$

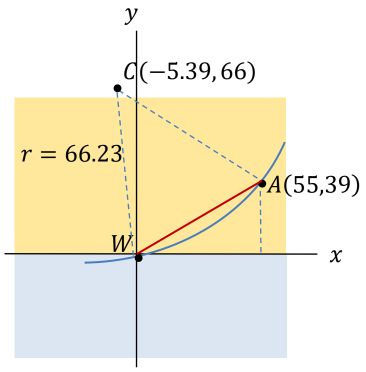
That is, the circle $\left(x + \frac{1}{a}\sqrt{\rho^2 - {v_{\max}}^2}\right)^2 + \left(y - \frac{1}{a}v_{\max}\right)^2 = \left(\frac{\rho}{a}\right)^2$.

Given the point (x_A, y_A) and values for a and b, it is possible to solve the equation for ρ. We find, $\rho^2 = \left(\frac{a}{2x}\right)^2 \left(x^2 + y^2 - \frac{2y}{a}v_{\max}\right)^2 + {v_{\max}}^2$.

An example with numbers and times

Let the perpendicular distance from the lifeguard station to the water be $d = 55$ m and let the point W be located 39 m along the water's edge to the left of the lifeguard station. The runner's speed varies from $v_{\min} = 1$ m/s to $v_{\max} = 6$ m/s. Thus, we have $a = \frac{1}{11}$ and we calculate $\rho = 6.02$ and therefore, $r = 66.23$ m.

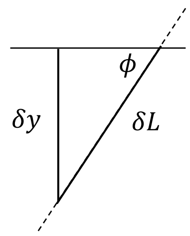
Thus, the least-time path is a circular arc with its centre located approximately at a point $(-5.39, 66)$.



The times taken along the circular arc and along a direct line with the same terminal point can be compared.

Time along line

The length of the chord AW is $\sqrt{39^2 + 55^2} \approx 67.42$ m.



The time over a small element of length $\delta L = \frac{\delta y}{\sin\phi}$ depends on its position on the line.

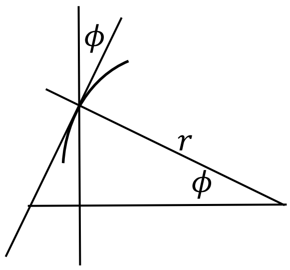
And, $\sin\phi = \frac{39}{67.42} \approx 0.5784$. Since $\delta t = \frac{\delta L}{v}$, we have $\delta t = \frac{\frac{\delta y}{0.5784}}{a(d-y)+b} = \frac{\frac{\delta y}{0.5784}}{v_{\max}-ay} \approx \frac{19.017}{66-y}\delta y$.

We form the differential equation $\frac{dt}{dy} = \frac{19.017}{66-y}$ and, since $0 \le y \le 39$, we obtain $t = [-19.017\ln(66-y)]_0^{39}$. Thus, the time for the journey along the line is approximately 17 s, with an average speed of 3.97 m/s.

Time along arc

For the arc, the angle at the centre is $\cos^{-1}\left(1 - \frac{55^2+39^2}{2\times 66.23^2}\right) \approx 1.068$ so that the arc length is approximately 70.74 m.

For the arc, we have $\delta t = \frac{\delta L}{v}$. Hence, $\delta t = \frac{r\,\delta\phi}{\rho \sin\phi}$.



and we form the equation $\frac{dt}{d\phi} = \frac{11}{\sin\phi}$. Integration gives $t = 11\ln\left(\tan\frac{\phi}{2}\right) + C$. At point A, $t = 0$ and we have $\tan\phi = \frac{66-39}{55+5.39} = 0.447$ and therefore $\phi = 0.42$.

Similarly, at point W we have $\tan\phi = \frac{66}{5.39} = 12.245$ so that $\phi = 1.49$.

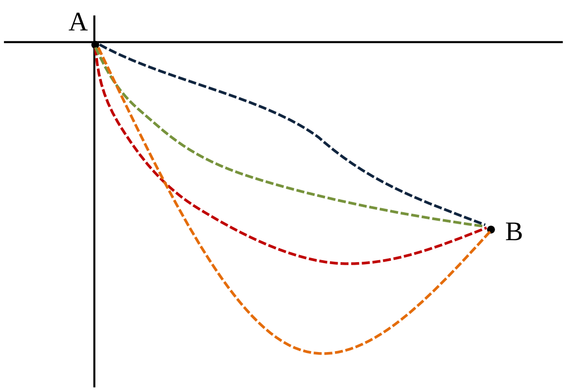
Thus, $t = \left[11\ln\left(\tan\frac{\phi}{2}\right)\right]_{0.42}^{1.49} \approx 16.1$ s. with an average speed of 4.4 m/s.

As expected, the arc is a longer but quicker path than the line.

The brachistochrone

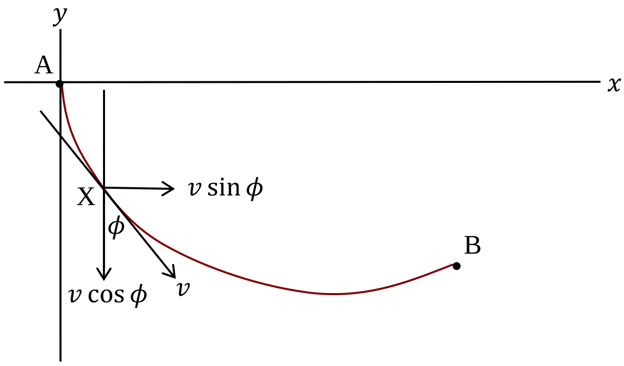
A question was posed by Johann Bernoulli in 1696 to the mathematicians of the day. Paraphrasing slightly, Bernoulli's problem is to find the path of least time taken by a bead sliding without friction on a wire under gravity between two points in a vertical plane.

Five mathematicians of the time are known to have provided solutions: Isaac Newton, Jakob Bernoulli, Gottfried Leibniz, Ehrenfried Walther von Tschirnhaus and Guillaume de l'Hôpital [4][5]. Newton claimed to have worked out his solution in one evening.



Johann Bernoulli saw a connection between his problem and the path of a beam of light refracted through a medium of variable density. We assume the problem is a special case of the lifeguard scenario.

The inertia of a mass accelerating due to gravity is like the soft sand through which the lifeguard runs; and it seems reasonable to liken the force of gravity to the effort exerted by the runner.



Bernoulli's problem and that of the runner on the beach differ in the way the speed of the moving object varies and in the way the endpoint B is approached.

Equation (2), $v = \rho \sin\phi$, is assumed to hold, as it did for the lifeguard, and so do the equations $\delta y = v\sqrt{1-\left(\frac{v}{\rho}\right)^2}\,\delta t$, $\delta x = \frac{v^2}{\rho}\delta t$ and equation (3) but the quantity ρ is determined differently.

An appropriate speed function v is needed.

Equations of motion

In physics, one writes $a = \frac{dv}{dt}$, expressing an acceleration a as the rate of change of a velocity v with respect to time t. Hence, $\int a\,dt = \int dv$ and so, when a is a constant, we have

$$v = at + u \qquad (M1)$$

where u is the velocity at time $t = 0$.

Similarly, $v = \frac{ds}{dt}$ expresses velocity as the time rate of change of displacement s. Therefore, $\frac{ds}{dt} = at + u$ and on taking antiderivatives we have

$$s = ut + \frac{1}{2}at^2 \qquad (M2)$$

provided $s = 0$ at $t = 0$.

By making t the subject of equation (1) and substituting this expression for t in equation (2), we obtain

$$v^2 = u^2 + 2as \qquad (M3).$$

A falling object is accelerating at a constant rate, $-g$, due to the constant gravitational force. Its speed at a point X on the curve depends on its distance below the initial point A. Indeed, a consideration of the kinetic energy of the object shows that its speed is just the magnitude of the velocity it would have attained if falling vertically.

According to equation (3), the speed must be

$$v = \sqrt{2gy}$$

where y is the distance below point A.

As before, we can form two equations: $\delta y = v\sqrt{1 - \left(\frac{v}{\rho}\right)^2}\,\delta t$ and $\delta x = \frac{v^2}{\rho}\delta t$. In the lifeguard example we formed an expression for the derivative $\frac{dy}{dx}$ from these equations. However, in this case the procedure is unedifying. Instead, from the equation for δy, and with the substitution $v = \sqrt{2gy}$, we form and solve

$$\frac{dy}{dt} = \sqrt{2gy}\sqrt{1 - \left(\frac{\sqrt{2gy}}{\rho}\right)^2}.$$

Hence, $t = \frac{\rho}{g}.\sin^{-1}\frac{\sqrt{2gy}}{\rho}$, since the constant of integration is zero.

Therefore, $\sin\frac{gt}{\rho} = \frac{\sqrt{2gy}}{\rho}$ and so, after rearrangement, $y = \frac{\rho^2}{2g}\sin^2\left(\frac{gt}{\rho}\right)$. Using a cosine double angle formula, we rewrite this as

$$y = \frac{\rho^2}{4g}\left(1 - \cos\frac{2gt}{\rho}\right) \qquad (4)$$

Then, from $\delta x = \frac{v^2}{\rho}\delta t$ and $v = \sqrt{2gy}$, we form $\frac{dx}{dt} = \frac{2gy}{\rho}$, and therefore, using (4) to substitute for y, we have

$$\frac{dx}{dt} = \frac{\rho}{2}\left(1 - \cos\frac{2gt}{\rho}\right).$$

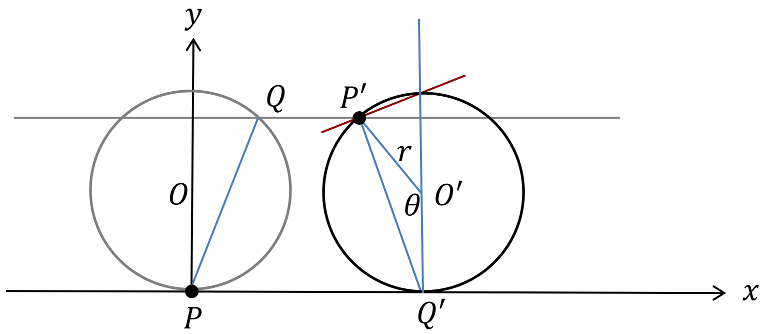
This differential equation has the solution $x = \frac{\rho}{2}\left(t - \frac{\rho}{2g}\sin\frac{2gt}{\rho}\right)$. (Again, the constant of integration is zero.)

Thus, in parametric form, the least-time path is given by

$$x(t) = \frac{\rho^2}{4g}\left(\frac{2gt}{\rho} - \sin\frac{2gt}{\rho}\right)$$
$$y(t) = \frac{\rho^2}{4g}\left(1 - \cos\frac{2gt}{\rho}\right) \qquad (5)$$

Cycloid

We recognise equations (5) as those describing a cycloid, the locus of a point on a circle rolling on a line. The following diagram illustrates the situation.



If the rotation angle $P'O'Q'$ is θ, and r is the radius of the generating circle, then the distance PQ' is equal to the arc length $P'Q' = r\theta$ and it follows that the cycloid is given in parametric form by

$$x = r(\theta - \sin\theta)$$
$$y = r(1 - \cos\theta)$$

References

[1] https://en.wikipedia.org/wiki/Snell%27s_law#Derivations_and_formula

[2] Fermat, Pierre de (1638). *Methodus ad Disquirendam Maximam et Minimam*.

[3] Huygens, Christiaan (1678). *Traite de la lumiere*.

[4] https://en.wikipedia.org/wiki/Brachistochrone_curve

[5] https://en.wikipedia.org/wiki/Brachistochrone_curve#Johann_Bernoulli's_solution