FIBONACCI *A compilation*

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Origin

The book *Liber abaci* completed in 1202 by Leonardo of Pisa (known as Fibonacci), contains the challenge: *Calculate how many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on.* (Boyer, 1989)

The question gives rise to the Fibonacci sequence.

$$F_n = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ F_{n-2} + F_{n-1} & \text{if } n > 2 \end{cases}$$

Written explicitly, the sequence is $(1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots)$.

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Related sequences

Given its two-step recursion rule, the Fibonacci sequence is completely determined by the first two terms. A different choice of the initial terms leads to a different sequence. One such is known as the sequence of Lucas numbers (an example of a more general Lucas sequence), after François Édouard Anatole Lucas (1842–91). It is defined analogously to the Fibonacci sequence by

$$L_n = \begin{cases} 2 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ L_{n-2} + L_{n-1} & \text{if } n > 2 \end{cases}$$

with the explicit representation $(2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \dots)$.

Many sequences of this kind are possible. However, since each of them is determined by its initial two terms they can all be considered as linear combinations of an independent pair of such sequences. For example, in the vector space of integer sequences with the same recursive rule that produces the Fibonacci and Lucas numbers, the sequences $(s_n) = (1, 0, 1, 1 \dots)$ and $(t_n) = (0, 1, 1, 2, \dots)$ form a basis because any pair of initial terms (a, b) can be expressed as a linear combination of the initial terms $(1,0)$ and $(0,1)$. Thus,

$$F_n = s_n + t_n$$

and

$$L_n = 2s_n + t_n.$$

We might consider sequences generated from their first two terms but with a different recursion rule. That is, $a_{n+1} = Aa_n + Ba_{n-1}$ given constants A and B and particular values of the terms a_1 and a_2. Again, sequences beginning $(1,0, \dots)$ and $(0,1, \dots)$ form a basis for the space of sequences that are defined by a choice of recursion rule.

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Lucas sequences

There is a notation that indicates both a recursion rule and a standard set of initial terms that produce a basis pair of sequences. The notation gives *Lucas sequences* of the first and second kind, $U(P, Q)$ and $V(P, Q)$ respectively.

The recurrence relation is assumed to be $x_n = Px_{n-1} - Qx_{n-2}$, with $n \in \{0, 1, 2, \dots\}$. Then, $U(P, Q)$ is defined by

$$U_n(P,Q) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ P \cdot U_{n-1} - Q \cdot U_{n-2} & n > 1 \end{cases}$$

and $V(P,Q)$ is defined by

$$V_n(P,Q) = \begin{cases} 2 & n = 0 \\ P & n = 1 \\ P \cdot V_{n-1} - Q \cdot V_{n-2} & n > 1 \end{cases}$$

The sequence $U_n(1,-1)$ is the Fibonacci sequence, and the sequence $V_n(1,-1)$ corresponds to the Lucas numbers.

Properties of the Fibonacci sequence often follow from properties observed in this more general setting.

Another generalisation of the Fibonacci-type sequence is obtained by making the recursive definition depend on three or more successive terms rather than two. For example, with three terms,

$$T_n = \begin{cases} a & \text{if } n = 1 \\ b & \text{if } n = 2 \\ c & \text{if } n = 3 \\ T_{n-3} + T_{n-2} + T_{n-1} & \text{if } n > 3 \end{cases}$$

If $a = b = 0$ and $c = 1$ the sequence is: $(0, 0, 1, 1, 2, 4, 7, 13, 24, 44, \dots)$. Three-step sequences of this kind are called *tribonacci sequences*.

The family of three-step sequences is generated as linear combinations of sets of three basis sequences, for example $(0, 0, 1, \dots)$, $(0, 1, 0, \dots)$ and $(1, 0, 0, \dots)$.

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Some identities

Given the Fibonacci sequence (F_n), we have by definition,

$$F_{n+1} = F_{n-1} + F_n$$

and then

$$F_{n+2} = F_n + F_{n+1} = F_{n-1} + 2F_n$$

Adding these equations gives

$$F_{n+3} = 3F_n + 2F_{n-1} = F_4 F_n + F_3 F_{n-1}$$

which suggests that by adding successive pairs of equations, we should obtain

(1) $\quad F_{n+k} = F_{k+1}F_n + F_k F_{n-1}$

This is proved inductively by assuming that the statements for F_{n+k} and F_{n+k-1} are true.

Setting $k = n$ gives

(2) $\quad F_{2n} = F_n(F_{n+1} + F_{n-1})$

Setting $k = n - 1$ in (1) gives

(3) $\quad F_{2n-1} = {F_n}^2 + {F_{n-1}}^2$

Setting $k = n + 1$ in (1) gives

(4) $\quad F_{2n+1} = F_{n+2}F_n + F_{n+1}F_{n-1}$

By manipulating (4) or by induction

(5) $\quad F_{2n+1} = F_{n+2}F_{n+1} - F_nF_{n-1}$

(6) $\quad F_{n+2}F_n - {F_{n+1}}^2 = (-1)^{n+1}$

Identity (6) was discovered in 1680 by Giovanni Cassini. To verify (6), consider a two-step recursive process $a_{n+1} = a_n + a_{n-1}$.
Let $N_k = a_{k+1}a_{k-1} - {a_k}^2$. Then, using $a_{k-1} = a_{k+1} - a_k$, we have

$$\begin{aligned} N_k &= a_{k+1}(a_{k+1} - a_k) - {a_k}^2 \\ &= {a_{k+1}}^2 - a_k(a_{k+1} + a_k) \\ &= -(a_{k+2}a_k - {a_{k+1}}^2) \\ &= -N_{k+1} \end{aligned}$$

In the case of the Fibonacci numbers, observe that when $k = 2$, $N = F_3F_1 - {F_2}^2 = 1$, and when $k = 3$, $N = -1$. Formula (6) follows by induction.

Another proof (Knuth, 1997) uses a matrix. Consider powers of the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.
$A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$, … and, $A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$. Taking the determinants of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$ and $\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$, we have $(-1)^n = F_{n+1}F_{n-1} - {F_n}^2$.

An extension of Cassini's identity, discovered in 1879 by Eugène Charles Catalan, says:

(7) $\quad {F_n}^2 - F_{n+k}F_{n-k} = (-1)^{n-k}{F_k}^2$.

Given a more general sequence (S_n) defined by $S_n = \begin{cases} 1 & \text{if } n = 1 \\ q & \text{if } n = 2 \\ qS_{n-1} + S_{n-2} & \text{if } n > 2 \end{cases}$, we show by induction that

(8) $\quad S_{k+1}S_{n-k} + S_kS_{n-(k+1)} = S_n$ for $k \in \{1, 2, \dots, n-2\}$

To begin, when $k = 1$, and $n > 2$, we have $S_2S_{n-1} + S_1S_{n-2} = qS_{n-1} + S_{n-2} = S_n$. Suppose the statement is true for some arbitrary $k \in \{1, 2, \dots, r-2\}$. Then,

$$\begin{aligned} S_n &= S_{k+1}S_{n-k} + S_kS_{n-(k+1)} \\ &= S_{k+1}\left(qS_{n-(k+1)} + S_{n-(k+2)}\right) + S_kS_{n-(k+1)} \\ &= qS_{k+1}S_{n-(k+1)} + S_{k+1}S_{n-(k+2)} + S_kS_{n-(k+1)} \\ &= S_{n-(k+1)}(qS_{k+1} + S_k) + S_{k+1}S_{n-(k+2)} \\ &= S_{k+2}S_{n-(k+1)} + S_{k+1}S_{n-(k+2)}. \end{aligned}$$

That is, the statement remains true when $k + 1$ is replaced by $k + 2$, and the conclusion follows.

When $n = 2r$ and $k = r$, we have $S_{2r} = S_{r+1}S_r + S_rS_{r-1}$, which is equivalent to (2) in the Fibonacci case, $q = 1$.

When $n = 2r - 1$ and $k = r - 1$, we have $S_{2r-1} = {S_r}^2 + {S_{r-1}}^2$, which reduces to (3) when $q = 1$.

Cassini's identity appears as $\det\begin{pmatrix} q & 1 \\ 1 & 0 \end{pmatrix}^n = \det\begin{pmatrix} S_{n+1} & S_n \\ S_n & S_{n-1} \end{pmatrix}$, so that $(-1)^n = S_{n+1}S_{n-1} - {S_n}^2$.

(9) $\varphi^n = \varphi F_n + F_{n-1},\ n \geq 2$, where $\varphi = \frac{1+\sqrt{5}}{2}$. (See below for the significance of φ.)

This is proved by induction using the fact that $\varphi^2 = \varphi + 1$. The statement is clearly true when $n = 2$ since the statement $\varphi^2 = \varphi F_2 + F_1$ is just $\varphi^2 = \varphi + 1$. If it is true for some k, then on multiplying both sides by φ we have $\varphi^{k+1} = \varphi^2 F_k + F_{k-1} = (1+\varphi)F_k + F_{k-1} = \varphi F_k + F_{k+1}$, which is the inductive step.

A more direct explanation for this identity is given below.

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Ratios of successive terms

Ratios of successive terms of the Fibonacci sequence form a sequence:

$$\left(\frac{F_{k+1}}{F_k}\right)_{k=1}^{\infty} = \left(1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots\right).$$

This is an alternating sequence with the odd numbered terms forming an increasing subsequence and the even numbered terms forming a decreasing subsequence.

The difference between successive terms of the ratio sequence is given by $\frac{F_{k+1}}{F_k} - \frac{F_k}{F_{k-1}} = \frac{F_{k+1}F_{k-1} - {F_k}^2}{F_k F_{k-1}}$. According to identity (6) above, this expression is equal to $\frac{(-1)^{k-1}}{F_k F_{k-1}}$ which approaches 0 as $k \to \infty$. Therefore, the sequence of ratios converges. The limit, usually notated φ, is found by setting

$$\begin{aligned}\varphi &= \lim_{k\to\infty.} \frac{F_{k+1}}{F_k} \\ &= \lim_{k\to\infty.} \frac{F_{k-1} + F_k}{F_k} \\ &= \lim_{k\to\infty.} \frac{F_{k-1}}{F_k} + 1 \\ &= \frac{1}{\varphi} + 1\end{aligned}$$

Therefore, $\varphi^2 - \varphi - 1 = 0$, with positive solution $\varphi = \frac{1+\sqrt{5}}{2}$.

More generally, in any sequence generated by $a_{n+1} = a_n + a_{n-1}$, whatever the initial terms, the sequence of ratios of successive terms is shown to converge (by the same reasoning as was used in the Fibonacci case) and the limit is the same number, φ.

If the recursion is $a_{n+1} = Aa_n + Ba_{n-1}$, we find (assuming it exists) the limit

$$\lambda = \lim_{k\to\infty.} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty.} \frac{Aa_{k-1} + Ba_k}{a_k} = \frac{A}{\lambda} + B$$

Then, on solving $\lambda^2 - B\lambda - A = 0$, we obtain the positive root $\lambda = \frac{1}{2}\left(B + \sqrt{B^2 + 4A}\right)$.

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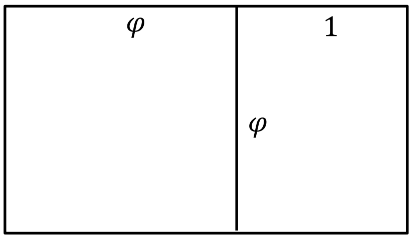
Phi φ

As well as being the limit of the sequence of Fibonacci ratios, the number $\frac{1+\sqrt{5}}{2}$ is known as the 'golden ratio', the 'golden section', the 'divine proportion', and more recently, 'phi'. Rectangles with sides in this proportion are considered by artists and architects to be aesthetically pleasing.



In the diagram above, the larger rectangle is divided so that the smaller rectangle is similar to it. That is, $\frac{\varphi}{1} = \frac{\varphi+1}{\varphi}$.

According to Theodore Andrea Cook (Wikipedia), engineer Mark Barr (1871-1950) gave the golden ratio, the number $\frac{1+\sqrt{5}}{2}$, the name *phi* (φ). Cook wrote that Barr chose φ by analogy to the use of π for the ratio of a circle's circumference to its diameter and because it is the first letter in the name of the ancient Greek sculptor Phidias.

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A two-step sequence built from φ

The fact that successive terms of the Fibonacci sequence approach a constant ratio means the Fibonacci sequence approaches a geometric sequence.

Consider a recurrence relation given by $a_n = a_{n-1} + a_{n-2}$ with initial terms 1 and φ. The resulting sequence is $(f_n) = (1, \varphi, 1+\varphi, 1+2\varphi, 2+3\varphi, 3+5\varphi, \dots)$ with general term $f_n = F_{n-2} + F_{n-1}\varphi$. Since $\varphi^2 - \varphi - 1 = 0$, we have the terms

$1 + \varphi = \varphi^2$

Then,

$1 + 2\varphi = \varphi^2 + \varphi = \varphi(1+\varphi) = \varphi^3$

and then

$2 + 3\varphi = (1+\varphi) + (1+2\varphi) = \varphi^2 + \varphi^3 = \varphi^2(1+\varphi) = \varphi^4$

Continuing in this way we conclude that the sequence (f_n) is also the geometric sequence $(1, \varphi, \varphi^2, \varphi^3 \dots \varphi^n)$ with terms $f_n = \varphi^{n-1}$.

Identity (9) above follows from this discussion.

Using the standard geometric sum formula, we find $\sum_{n=1}^{k} f_n = \varphi^{k+1} - \varphi$.

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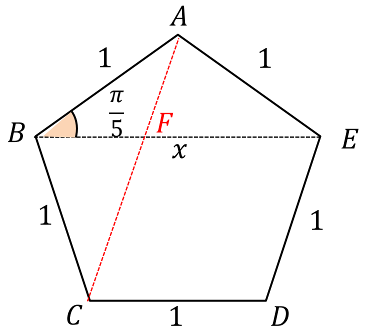
Further appearances of φ

The number φ occurs in contexts other than the Fibonacci sequence whenever the quadratic equation $x^2 - x - 1 = 0$ appears. In an example from the geometry of the regular pentagon, we obtain the exact value $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$, via the emergence of φ.



Let x be the length of the diagonal BE on the regular unit pentagon $ABCDE$. By the cosine rule on triangle ABE we have $1 = 1 + x^2 - 2x\cos\frac{\pi}{5}$ and thus $x = 2\cos\frac{\pi}{5}$.

Since BE is parallel to CD and AD is parallel to DE then $CDEF$ is a unit rhombus. Thus, by symmetry $AF = BF$ and we have $\Delta ABE \sim \Delta BAF$. In particular, $\frac{x}{1} = \frac{1}{x-1}$. This simplifies to $x^2 - x - 1 = 0$ so that $x = \varphi$. Then, from $\varphi = 2\cos\frac{\pi}{5}$, we deduce $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$.

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Explicit formulas for Fibonacci terms

Approximations

The Fibonacci sequence is not a geometric sequence but since the ratio between its successive terms approaches a constant, it is possible to imagine a geometric sequence whose terms are close to the corresponding terms of the Fibonacci sequence for terms far enough along the sequence. Thus, one idea would be to set $g_n = a\varphi^{n-1} \approx F_n$, and look for a suitable initial term a of the geometric sequence (g_n). (Another possibility is to set $h_{n+1} = b\varphi^n \approx F_n$ so that $b = \frac{F_n}{\varphi^n}$ is close to $\frac{1}{\sqrt{5}}$).

In the case of (g_n), $a \approx \frac{F_n}{\varphi^{n-1}}$. Then, candidate values of a, starting from $n = 5$, are (to a few decimal places) 0.729, 0.721, 0.724, 0.723, 0.7237, … and for $n \geq 12$, $a \approx 0.7236$. Thus,

$$F_n \approx g_n = 0.7236\varphi^{n-1}$$

The actual value of F_n is the nearest integer to the value of g_n given by this formula.

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Exact formula

A precise formula for terms of a sequence arising from a recurrence relation $a_{n+2} = Aa_{n+1} + Ba_n$ begins from the assumption that there exists a number α such that $a_n = \alpha^n$ for all positive integers n. Then, α satisfies $\alpha^2 = A\alpha + B$. In the case of the Fibonacci sequence, $A = B = 1$ so that the quadratic equation to be solved is $x^2 - x - 1 = 0$. (This is called the *characteristic equation.*)

There are two solutions, $x = \frac{1+\sqrt{5}}{2}$ and $x = \frac{1-\sqrt{5}}{2}$. These are often written φ^+ and φ^- respectively. It can be shown (Anderson, 1989) that any linear combination of the solutions must also satisfy the recurrence relation. Thus, the set of all possible solutions is expressed by $F_n = K_1(\varphi^+)^n + K_2(\varphi^-)^n$ with K_1 and K_2 being determined by the initial terms of the recurrence relation. In this case,

$F_1 = 1 = K_1\varphi^+ + K_2\varphi^-$ and
$F_2 = 1 = K_1(\varphi^+)^2 + K_2(\varphi^-)^2$

After multiplying the first of these equations by φ^+, and making use of the fact that $\varphi^+\varphi^- = -1$, we have $\varphi^+ = K_1(\varphi^+)^2 - K_2$. Then,

$1 - \varphi^+ = K_2((\varphi^-)^2 + 1)$ and therefore, $K_2 = \frac{1-\varphi^+}{(\varphi^-)^2+1} = -\frac{1}{\sqrt{5}}$. Then, $K_1 = \frac{\varphi^+ - \frac{1}{\sqrt{5}}}{(\varphi^+)^2} = \frac{1}{\sqrt{5}}$.

Hence,

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Since $\sqrt{5} = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}$ we can write $F_n = \frac{(\varphi^+)^n - (\varphi^-)^n}{\varphi^+ - \varphi^-}$. More generally, if α and β are respectively the positive and negative roots of a characteristic equation belonging to a Lucas sequence (S_n), then

$$S_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

This is known as Binet's formula.

Observe that when n is large, $\left(\frac{1-\sqrt{5}}{2}\right)^n \to 0$ so that F_n approaches the approximation $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$ mentioned above.

Another representation arises from the fact that $\varphi^- = -\frac{1}{\varphi^+}$, so that

$$F_n = \begin{cases} \frac{1}{\sqrt{5}}\left(\phi^n - \frac{1}{\phi^n}\right), & n \text{ even} \\ \frac{1}{\sqrt{5}}\left(\phi^n + \frac{1}{\phi^n}\right), & n \text{ odd} \end{cases}$$

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N-step sequences

Tribonacci

The sequence of ratios of successive terms in the three-step sequence

$$(T_n) = (0, 0, 1, 1, 2, 4, 7, 13, 24, 44, \dots)$$

also seems to converge to a limit. Assuming the limit exists, we write

$$\begin{aligned} \tau &= \lim_{k\to\infty.} \frac{T_{k+1}}{T_k} \\ &= \lim_{k\to\infty.} \frac{T_k + T_{k-1} + T_{k-2}}{T_k} \\ &= 1 + \lim_{k\to\infty.} \left(\frac{T_{k-1}}{T_k} + \frac{T_{k-2}}{T_k} \cdot \frac{T_{k-1}}{T_{k-1}}\right) \\ &= 1 + \frac{1}{\tau}\left(1 + \frac{1}{\tau}\right). \end{aligned}$$

Therefore, the limiting ratio τ satisfies the cubic equation $x^3 - x^2 - x - 1 = 0$.

Tetrabonacci

In a similar way, a four-step recursion leads to the sequence

$$(\mathrm{T}_n) = (0, 0, 0, 1, 1, 2, 4, 8, 15, 29, 56, \dots)$$

Again, the ratios $\frac{\mathrm{T}_{n+1}}{\mathrm{T}_n}$ appear to tend to a limiting value. By an argument similar to the three-step case, this can be shown to satisfy the fourth-degree polynomial equation

$$x^4 - x^3 - x^2 - x - 1 = 0$$

N-bonacci

The idea extends to an *n*-step recursion after the model of the Fibonacci sequence. The ratios of successive terms tend to a limit satisfying $x^n - x^{n-1} - \cdots - x^2 - x - 1 = 0$.

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Ratio limits—another expression

Suppose $x = \alpha$ satisfies $x^n - x^{n-1} - \cdots - x^2 - x - 1 = 0$. We show that α also satisfies a more concise equation that leads to further results. Note that α is a positive real number, greater than 1.

The equation $\alpha^n - \alpha^{n-1} - \cdots - \alpha^2 - \alpha - 1 = 0$ can be rewritten

$$\alpha^n - (\alpha^{n-1} + \cdots + \alpha^2 + \alpha + 1) = 0$$

Then,

$$\begin{aligned}
\alpha^n - \frac{(\alpha - 1)(\alpha^{n-1} + \cdots + \alpha^2 + \alpha + 1)}{\alpha - 1} &= 0 \\
\frac{\alpha^n(\alpha - 1) - (\alpha - 1)(\alpha^{n-1} + \cdots + \alpha^2 + \alpha + 1)}{\alpha - 1} &= 0 \\
\frac{\alpha^{n+1} - \alpha^n - (\alpha^n - 1)}{\alpha - 1} &= 0 \\
\frac{1}{\alpha - 1}(\alpha^{n+1} - 2\alpha^n + 1) &= 0
\end{aligned}$$

Now, $\frac{1}{\alpha-1} \neq 0$. Therefore,

$$\begin{aligned}
\alpha^{n+1} - 2\alpha^n + 1 &= 0 \\
\alpha^n(\alpha - 2) + 1 &= 0 \\
\alpha + \frac{1}{\alpha^n} &= 2
\end{aligned}$$

Hence, α satisfies the equation $x + \frac{1}{x^n} = 2$.

It follows that $\alpha = 2 - \frac{1}{\alpha^n}$ is less than 2, and α approaches 2 as $n \to \infty$.

As an historical note, Stephen Barr, whose father Mark Barr gave *phi* its name, sent Martin Gardner (Gardner, 1981) a clipping of an article by his father Mark Barr (from the London *Sketch* circa. 1913) in which he stated that α and n were related by the equation

$$n = \frac{\log(2 - \alpha)^{-1}}{\log \alpha}$$

Rearrangement of this equation gives

$$\alpha^{n+1} - 2\alpha^n + 1 = 0$$

consistent with the result given above.

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Fibonacci divisors of Fibonacci numbers

F_n divides F_{mn} for all $m \geq 2$

Proof

From identity (2), $F_{2n} = F_n(F_{n+1} + F_{n-1})$, it is clear that F_n divides F_{2n}. If it is true that $F_n|F_{kn}$ for some k, then $F_{(k+1)n} = F_{kn+n}$ and by identity (1) this is $F_{n+1}F_{kn} + F_nF_{kn-1}$so that $F_n|F_{(k+1)n}$. Then, since $F_n|F_{kn}$ when $k = 2$, the conclusion follows by induction.

Sequences with this property are called *divisibility sequences*.

Greatest common divisor

For positive integers m and n with greatest common divisor (m, n), the greatest common divisor of Fibonacci numbers F_m and F_n is $F_{(m,n)}$.

Proof

The procedure is essentially the Euclidean algorithm for the greatest common divisor of two numbers. We make use of identity (1), $F_{n+k} = F_{k+1}F_n + F_kF_{n-1}$.

If $m > n$ we can write $m = n + k$. Any divisor of m and n is a divisor of k. And, any common divisor of F_m and F_n divides $F_m = F_{n+k} = F_{k+1}F_n + F_kF_{n-1}$. Such a divisor does not divide F_{n-1} and therefore must divide F_k, which is less than F_m.

Thus, $(F_m, F_n) = (F_k, F_n)$. If $k = n$ then $(F_m, F_n) = F_n$. Otherwise, $k \neq n$ and the reduction process can be repeated in a similar way. For example, if $k > n$, we have $k = n + s$ and any divisor of n and k also divides s. It follows, as before, that $(F_k, F_n) = (F_s, F_n), s < k$. Eventually, either a subscript 1 is reached or the two subscripts are equal, and the process terminates. Either

$(F_m, F_n) = (F_k, F_n) = \cdots = (F_1, F_r) = 1$ or

$(F_m, F_n) = (F_k, F_n) = \cdots = (F_r, F_r) = F_r$.

The Euclidean procedure applied to the subscripts alone leads to the greatest common divisor of m and n. Therefore, both $(F_m, F_n) = F_r$ and $(m, n) = r$. Hence, $(F_m, F_n) = F_{(m,n)}$.

Sequences with this property are called *strong divisibility sequences*.

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Integer divisors of Fibonacci numbers

Insights from modular arithmetic

Every third Fibonacci number is even: the others are odd. The sequence expressed in modulo 2 is, $(1, 1, 0, 1, 1, 0, \ldots)$.

Similarly, every fourth Fibonacci number is divisible by 3, since in mod 3 we have the 8-digit repeating pattern $(1, 1, 2, 0, 2, 2, 1, 0, \ldots)$.

However, every sixth Fibonacci number is divisible by 4 and every fifth Fibonacci number is divisible by 5.

Pisano period

It is observed that the pattern of successive terms modulo n repeats. The length of the repeating pattern for each modulus is called its *Pisano period*. (Fibonacci was known as Pisano after his hometown, Pisa.)

In fact, in modulo n, the sequence of terms must repeat because in a finite string of symbols, some ordered pair of them must recur eventually and then the pattern begins again due to the recurrence rule. Given n symbols, there are n^2 cyclically ordered pairs, but the pair $(0, 0)$ must be excluded and some other pairs may fail to appear. Thus, the sequence of symbols repeats after at most $n^2 - 1$ terms.

In modulo 5, for example, the Fibonacci sequence is

$$(1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, \dots)$$

and its Pisano period is 20. Observe that the ordered pair $(2, 1)$ does not appear. If the recursion rule is applied to this absent pair, we generate the length 4 repeating pattern:

$$(2, 1, 3, 4, \dots)(\text{mod } 5)$$

and it includes the four ordered pairs missing from the $n^2 - 1$ estimate.

In modulo 10, the sequence includes all ten digits and must have a Pisano period of at most $n^2 - 1 = 99$. Ordered pairs belonging to strings containing the even digits exclusively do not occur. These are: $(2, 2, 4, 6, 0, 6, 6, 2, 8, 0, 8, 8, 6, 4, 0, 4, 4, 8, 2, 0)$ with length 20, and $(2, 6, 8, 4)$ with length 4. Also, the ordered pairs from the strings $(5, 5, 0)$ of length 3 and $(1, 8, 9, 7, 6, 3, 9, 2, 1, 3, 4, 7)$, length 12, are not found. Thus, in mod 10, the Pisano period is 60.

For every n there is a corresponding Pisano period $\pi(n)$. In each case the repeating string begins with $(1, 1, \dots)$and, hence, must include $(\dots, 1, 0, 1, 1, \dots)(\text{mod } n)$. Therefore, for each n there exists an F_n with n as a factor. In particular, every prime divides some Fibonacci number.

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Successive terms are coprime

In a two-step Fibonacci-type sequence, no term has a prime factor in common with its predecessor.

Suppose terms F_k and F_{k+1} do have a common prime factor for some value of k. Then, the preceding term $F_{k-1} = F_{k+1} - F_k$ must also have this factor. Similarly, each of the preceding terms down to $F_4 = 3$ and $F_3 = 2$ must share the factor. Since this assertion is false, it follows that F_k and F_{k+1} could not have had a common prime factor and, therefore, successive pairs of terms must be coprime.

Moreover, terms F_k and F_{k+2} are coprime. Otherwise, $F_{k+1} = F_{k+2} - F_k$ would have a factor in common with both F_k and F_{k+2}.

Similarly, every three successive terms in the sequences $(s_n) = (1, 0, 1, 1, \dots)$ and $(t_n) = (0, 1, 1, 2, \dots)$ are pairwise coprime as these sequences are the same as the Fibonacci sequence but with shifted starting points. (That is, $s_{k+2} = F_k$ and $t_{k+1} = F_k$) Linear combinations $(r_n) = a(s_n) + b(t_n)$ have the same property provided a and b are coprime.

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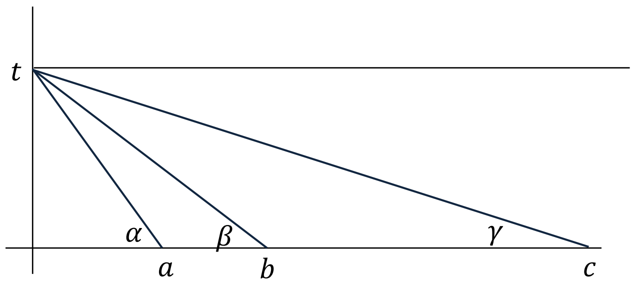
Powers of 2 and 3
The greatest power of 2 that can occur as a Fibonacci number is $2^3 = 8$. For, if there exists $F_k = 2^i, i > 3$, it is divisible by $1 = F_2$ and $2 = F_3$. The index $k > 3$ of the Fibonacci number F_k contains the factors 2 and 3 and no other distinct prime factors. If k has a second factor 2, then it has 4 as a factor and F_k is divisible by $F_4 = 3$ which is impossible. Similarly, if k has the factor 3 repeated, then it has 9 as a factor and F_k is divisible by $F_9 = 34$ which is also impossible.

If there exists $F_k = 3^i, i > 1$, then F_k is divisible by $1 = F_2$ and $3 = F_4$. The index k can have no prime factor other than 2. The prime factor 2 occurs twice in the index of F_4. If it should occur more than twice, then the index k must be divisible by 8. This is impossible because $F_8 = 21$, which has a divisor 7 that does not divide any power of 3.

Primes of form $4k + 1$
In a study of *Gardner triples* (Turner & Staples 2019) it is shown that a Fibonacci number F_{2n-1} can only be 1, a prime of the form $4k + 1$, a product of such primes or twice any of these quantities.

For certain integer values of a, b, c, t in the following diagram, it happens that the angles α, β, γ are such that $\alpha = \beta + \gamma$.



That is, $\tan^{-1}\frac{t}{a} = \tan^{-1}\frac{t}{b} + \tan^{-1}\frac{t}{c}$. Hence, $\frac{t}{a} = \frac{\frac{t}{b}+\frac{t}{c}}{1-\frac{t^2}{bc}}$. On rearrangement, we find $c = a + \frac{a^2+t^2}{b-a}$, which can only be an integer if $b - a$ divides the sum of squares $a^2 + t^2$. This occurs only if the difference $b - a$ is 1 or a prime of the form $4k + 1$ or a product of such primes or twice any of these quantities.

The condition $c = a + \frac{a^2+t^2}{b-a}$ is satisfied by

$$\tan^{-1}\frac{1}{F_{2n}} = \tan^{-1}\frac{1}{F_{2n+1}} + \tan^{-1}\frac{1}{F_{2n+2}}$$

Then, the result follows from the observation that the difference $b - a$ is $F_{2n+1} - F_{2n} = F_{2n-1}$.
Proof
From the Cassini identity (6) $F_{n+2}F_n - {F_{n+1}}^2 = (-1)^{n+1}$, we have $F_{2n+1}F_{2n-1} - {F_{2n}}^2 = 1$. Then

$$F_{2n+1}F_{2n-1} = {F_{2n}}^2 + 1$$

$$F_{2n+1} = \frac{{F_{2n}}^2 + 1}{F_{2n-1}}$$
$$F_{2n+2} - F_{2n} = \frac{{F_{2n}}^2 + 1}{F_{2n-1}}$$
$$F_{2n+2} = F_{2n} + \frac{{F_{2n}}^2 + 1}{F_{2n-1}}$$

and this corresponds with the form $c = a + \frac{a^2+t^2}{b-a}$ as required.

Primitives

A prime factor that has not occurred in any smaller number in the sequence is called a *primitive*.

R. D. Carmichael (Carmichael 1913) showed that except for F_1, F_2, F_6 and F_{12}, every Fibonacci number contains a primitive.

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Partial sums

From the Fibonacci sequence $(F_n) = (1, 1, 2, 3, 5, \dots)$, we have the sequence of partial sums $(S_n) = (1, 2, 4, 7, 12, \dots)$. By inspection, it appears that $S_n = F_{n+2} - 1$.

Proof

Assume the statement that $S_k = F_{k+2} - 1$ is true for some index k. Then we have the inductive step $S_{k+1} = S_k + F_{k+1} = F_{k+2} - 1 + F_{k+1} = F_{k+3} - 1$. Therefore, since it is true for $k = 1$, it must be true for all $n \in \mathbb{Z}^+$.

A general statement of a similar kind can be made in the case of two-step sequences with the same recurrence rule but different initial terms. In particular, consider the sequence

$$(t_n) = (a, b, a + b, a + 2b, 2a + 3b, \dots)$$

with the general term $t_n = F_{n-2}a + F_{n-1}b$.

The corresponding sequence of partial sums,

$$(s_n) = (a, a + b, 2a + 2b, 3a + 4b, 5a + 7b, \dots)$$

has the general term $s_n = F_n a + (F_{n+1} - 1)b$. This reduces to the previous result when $a = b = 1$.

Steve Thornton describes (Turner & Thornton 2017) a classroom exercise in which a student is asked to choose a pair of seed numbers and then to generate on a whiteboard the first ten terms of the sequence arising from the recurrence relation $a_{n+2} = a_n + a_{n+1}$. Other members of the class calculate the cumulative sum progressively. Mysteriously and unerringly, the teacher announces the sum of all ten terms of the sequence when only seven have been written down. The trick, which needs to be explained, lies in the fact that the tenth partial sum is always eleven times the seventh term.

Indeed, in the case of the sequence (F_n), we have $F_7 = 13$ while $S_{10} = F_{12} - 1 = 143$ and thus, $S_{10} = 11 \times F_7$ as claimed.

More generally, from the previous result we have $t_7 = F_5 a + F_6 b = 5a + 8b$ while $s_{10} = F_{10} a + (F_{11} - 1)b = 55a + 88b$. Again, $s_{10} = 11 \times t_7$, and this explains the classroom activity described above.

Further relations between terms of these sequences and certain elements of the corresponding sequences of partial sums exist. For example, it can be observed that the following equations hold:

$$\begin{aligned} s_2 &= t_3 \\ s_6 &= 4t_5 \\ s_{10} &= 11t_7 \\ s_{14} &= 29t_9 \\ s_{18} &= 76t_{11} \\ s_{22} &= 199t_{13}. \end{aligned}$$

This sequence of equations is summarised by

$$s_{4n-2} = c_n t_{2n+1}$$

where the sequence of coefficients is given by $c_n = \begin{cases} 1 & n = 1 \\ 4 & n = 2 \\ 3c_{n-1} - c_{n-2} & n > 2 \end{cases}$

This recurrence relation has an explicit solution involving the quantities φ^+ and φ^-, but it is more usefully expressed as

$$c_n = 2F_{2n} - F_{2n-1}$$

To verify the assertion that $s_{4n-2} = c_n t_{2n+1}$ it is necessary to prove the equivalent statement

$$F_{4n-2} a + (F_{4n-1} - 1)b = (2F_{2n} - F_{2n-1})(F_{2n-1} a + F_{2n} b).$$

This holds if both

$$F_{4n-2} = (2F_{2n} - F_{2n-1})F_{2n-1}$$

and

$$F_{4n-1} - 1 = (2F_{2n} - F_{2n-1})F_{2n}.$$

Identities (2), (5) and (6) can be used in inductive proofs of these statements.

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Partial sums of Fibonacci squares

The sum of the squares of the first n Fibonacci numbers is the product of $F_n F_{n+1}$. Clearly, $1^2 + 1^2 = 1 \times 2$ and $1^2 + 1^2 + 2^2 = 2 \times 3$, and $1^2 + 1^2 + 2^2 + 3^2 = 3 \times 5$. If it is assumed that the pattern continues up to the sum of the first k squares, that is,

$$\sum_{i=1}^{k} {F_i}^2 = F_k F_{k+1}$$

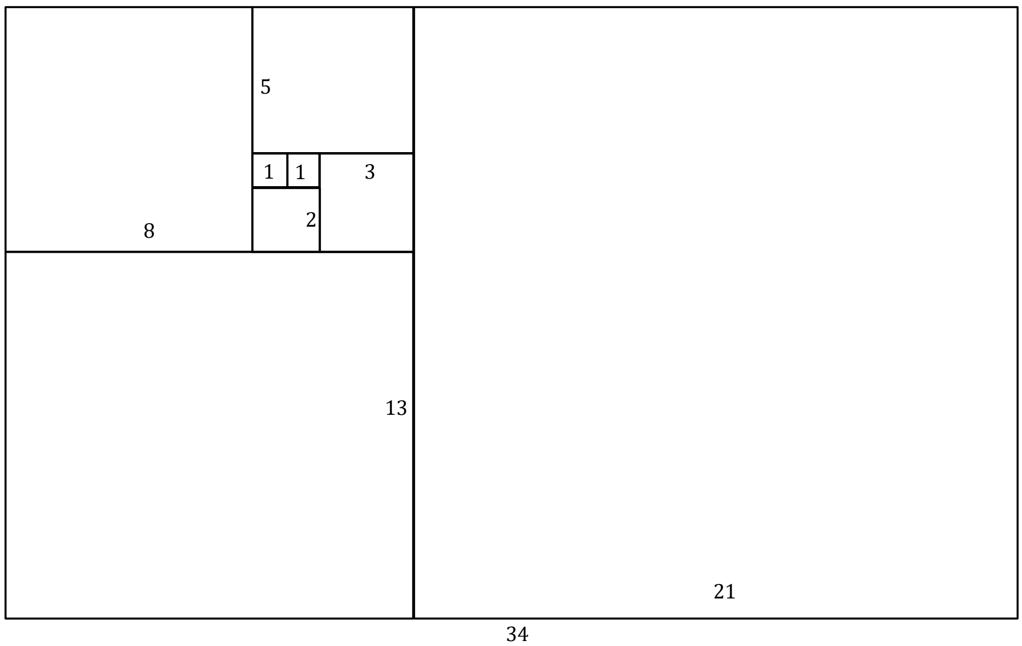
then adding the next square gives the inductive step

$$\sum_{i=1}^{k} {F_i}^2 + {F_{k+1}}^2 = F_k F_{k+1} + {F_{k+1}}^2 = F_{k+1}(F_k + F_{k+1}) = F_{k+1} F_{k+2}$$

It follows that

$$\sum_{i=1}^{n} {F_i}^2 = F_n F_{n+1} \quad \forall n \in \mathbb{Z}^+$$

However, the conclusion is visible in the following diagram. The area of the rectangle is both the sum of Fibonacci squares and the product of two Fibonacci numbers.



The diagram also shows that the Fibonacci squares tile the plane.

Almost-isosceles Pythagorean triples

Solutions (a, b, c) in positive integers of the equation $x^2 + y^2 = z^2$ are called *Pythagorean triples* due to their association with the sides of right triangles. It is well known that the countably infinite set of Diophantine solutions is given by

$$a = 2pq, b = p^2 - q^2, c = p^2 + q^2.$$

When a, b, c have no common factors, the solutions are called *primitive* Pythagorean triples. They occur when $p > q$, p and q have opposite parity, and p and q are coprime.

A non-empty subset of triples can be found such that a and b differ by 1. That is,

$$|(p^2 - q^2) - 2pq| = 1.$$

The triples $(3, 4, 5)$, $(20, 21, 29)$, and $(119, 120, 169)$, to name a few, are in this subset.

The quadratic equations implied by $|(p^2 - q^2) - 2pq| = 1$ have solutions for positive p,

$$p = q + \sqrt{2q^2 + 1} \qquad (1)$$

and

$$p = q + \sqrt{2q^2 - 1} \qquad (2)$$

It can be shown (Turner, 2006) that integer values of p come from the Lucas sequence $U(2, -1)$. Specifically, $(t_n) = (1, 2, 5, 12, 29, 70, \dots)$ with terms given by

$$t_n = 2t_{n-1} + t_{n-2}, \qquad t_1 = 1, t_2 = 2.$$

Equation (1) gives odd integer values for p when q is an even number in the sequence, while (2) gives even integer values when q is an odd term from the sequence. That is, beginning with $(1, 2, \dots)$ we can generate successive terms iteratively using equations (1) and (2) in alternation. Each pair of successive terms gives rise to an almost isosceles Pythagorean triple.

The sequence (t_n) has characteristic equation is $x^2 - 2x - 1 = 0$ with solutions $x = 1 \pm \sqrt{2}$. Thus, we deduce the explicit formula

$$t_n = \frac{\sqrt{2}}{4}\left[\left(1 + \sqrt{2}\right)^n - \left(1 - \sqrt{2}\right)^n\right].$$

For example, a very nearly isosceles Pythagorean triple arises from $t_{11} = 5741$ and $t_{12} = 13860$, so that

$$2t_{11}t_{12} = 149140520$$

and

$${t_{12}}^2 - {t_{11}}^2 = 149140519.$$

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Tiling a $1 \times n$ rectangle

The Fibonacci number F_{n+1} counts the number of ways in which a $1 \times n$ rectangle can be tiled with a collection of 1×1 and 1×2 tiles.

To see this, let $f(n)$ be the number of ways in which a $1 \times n$ rectangle can be tiled with a collection of 1×1 and 1×2 tiles. Either the final tile is a 1×1 tile, in which case there were $f(n-1)$ ways the other tiles could have been laid, or the final tile is a 1×2 tile and there were $f(n-2)$ ways in which the other tiles could have been laid. Thus, there are $f(n) = f(n-1) + f(n-2)$ ways in which n tiles can be laid. If $n = 2$, it is apparent that $f(2) = 2$ and if $n = 3$, then $f(3) = 3$. It follows that $f(4) = 5$ and $f(n) = F_{n+1}$.

Brent Yorgey (Yorgey, 2018) uses this fact to prove a theorem involving a sum of binomial coefficients. Consider the number of 1×2 tiles that could be chosen and, for each choice, the number of ways in which the tiles might be placed. If there are no 1×2 tiles, there are $\binom{n}{0}$ possibilities. If there is 1 1×2 tile, it can be placed in $\binom{n-1}{1}$ positions. If there are k 1×2 tiles they can be placed in $\binom{n-k}{k}$ places. In total, we see that

$$F_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{n-k}{k} + \cdots$$

By convention, the binomial coefficients are 0 if $k > n - k$. To obtain finite sums, we can have $k \geq \frac{n}{2}$ when n is even, and when n is odd, we require $k \geq \frac{n}{2} + \frac{1}{2}$.

So,

$$F_{2k+1} = \binom{2k}{0} + \binom{2k-1}{1} + \cdots + \binom{k}{k}$$

and

$$F_{2k} = \binom{2k-1}{0} + \binom{2k-2}{1} + \cdots + \binom{k}{k-1}.$$

Generating function

Sequences are often specified concisely as the coefficients of the terms of a polynomial. For example, the sum $S_\infty = x + x^2 + x^3 + \cdots$ generates the sequence 1, 1, 1, 1, … because the coefficients of the terms are all 1. On the other hand, provided $|x| < 1$, it is well-known that

$$S_\infty = \frac{x}{1-x}$$

The function $f(x) = \frac{x}{1-x}$ is said to be the generating function for the sequence 1, 1, 1, 1, … regardless of the values x can take.

The generating function for the Fibonacci sequence takes the form $A(x) = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots$. Then $xA(x) = x^2 + x^3 + 2x^4 + 3x^5 + 5x^6 + \cdots$, and $A(x) - xA(x) = x + x^3 + x^4 + 2x^5 + 3x^6 + \cdots$. Thus, $A(x)(1-x) = x + x^2A(x)$. On rearranging this, we obtain the concise expression

$$A(x) = \frac{x}{1 - x - x^2}.$$

The generating function can be used to deduce the explicit formula for Fibonacci terms. To achieve this, it is expanded again into a useful form. Note that the denominator $1 - x - x^2$ is the negative of the characteristic equation of the Fibonacci sequence, which has zeros $\varphi^+ = \frac{1+\sqrt{5}}{2}$ and $\varphi^- = \frac{1-\sqrt{5}}{2}$. Hence, the zeros of $1 - x - x^2$ are the negatives of these and therefore $A(x)$ can be written in the factorised form

$$A(x) = -\frac{x}{(x + \varphi^+)(x + \varphi^-)}.$$

By the technique of partial fractions this is, $A(x) = -\left[\frac{\frac{\varphi^+}{\sqrt{5}}}{x+\varphi^+} - \frac{\frac{\varphi^-}{\sqrt{5}}}{x+\varphi^-}\right]$, or using $\varphi^+\varphi^- = -1$,

$$A(x) = \frac{1}{\sqrt{5}}\left[\frac{1}{1 - x\varphi^+} - \frac{1}{1 - x\varphi^-}\right].$$

The formal quantity x can be chosen such that $|x\varphi^+| < 1$ and $|x\varphi^-| < 1$, so that the terms inside the bracket expand to the infinite series $1 + x\varphi^- + (x\varphi^-)^2 + \cdots$ and $1 + x\varphi^+ + (x\varphi^+)^2 + \cdots$ respectively. Putting the pieces together, we have

$$A(x) = \frac{1}{\sqrt{5}}[(\,x\varphi^+ + (x\varphi^+)^2, \dots) - (\,x\varphi^- + (x\varphi^-)^2, \dots)]$$

The nth term of the Fibonacci sequence is the coefficient of the nth term of $A(x)$, namely

$$F_n = \frac{1}{\sqrt{5}}[(\varphi^+)^n - (\varphi^-)^n]$$

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