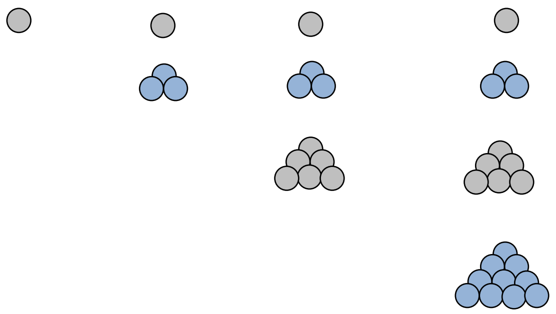
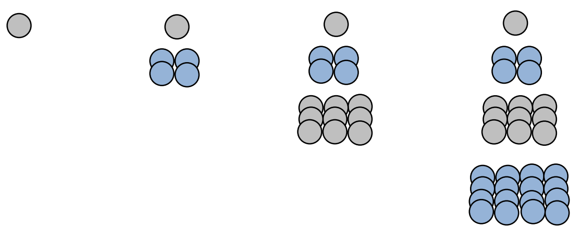
Sir Walter Raleigh's questions about cannon balls

In 1583, Thomas Harriot entered the service of Sir Walter Raleigh as an instructor in the art of navigation and as an expert in other mathematical matters. In about the year 1591, Raleigh asked Harriot some questions about stacking cannon balls, which Harriot was able to resolve.

- Given a number of cannon balls to be stacked above a triangular, square or rectangular base, how large should the base be?
- Given a pyramid of cannon balls, calculate the number of balls in the pile.

For an account of the life of Thomas Harriot, see
https://www-history.mcs.st-andrews.ac.uk/Biographies/Harriot.html





Two apparently different stacks of cannon balls are illustrated on the previous page, one with an equilateral triangle base and the other built on a square. In both arrangements, each layer of balls fits into the dimples between the balls in the layer below.

The stacks are certainly different if we want to calculate the numbers of balls involved but if we wish to know the ratio of the volumes of balls to voids in a large stack, or if we want to know how many neighbouring balls any ball inside the stack is touching, we find that that the packings are identical.

Counting tetrahedral numbers

Consider first the number of balls in a single layer of the stack. In the triangular pyramid arrangement, the nth layer contains the number of balls in the $(n-1)$th layer plus another n. For example, the third layer contains 6 balls, so the fourth layer contains $6+4=10$ balls. Thus, the numbers of balls in each successive layer forms the sequence of *triangular* numbers, $(1,3,6,10,15,\ldots)$. This is summarised by the formula $t_n = t_{n-1} + n$, where t_n is the nth triangular number and we agree that $t_1 = 1$.

This kind of formula describes a *recurrence relation*. It is also possible to give a formula for the triangular numbers that depends only on the number *n*. It is easy to check that the formula $t_n = \frac{n(n+1)}{2}$ gives the correct results, at least for the first few terms of the sequence of triangular numbers. We show how this formula can be derived and verified in the section 'Some recurrence relations'.

The triangular pyramid or *tetrahedron* is made up of layers of successive triangular numbers. The problem is that of adding successive triangular numbers. Again, the numbers of balls in successive tetrahedra can be described by a recurrence relation. We can write $T_n = T_{n-1} + t_n$ where T_n is the nth tetrahedral number. We agree that $T_0 = 0$ and $n = 1,2,3,\ldots$. As before, we can obtain a formula depending only on n for the tetrahedral numbers. We claim that

$$T_n = \frac{n}{6}(n+2)(n+1)$$

and the derivation for this can be done by methods similar to those described in the section on recurrence relations.

Counting square pyramidal numbers

In the square pyramid stack, the nth layer contains a square number of balls. That is, if s_n is the number of balls in the nth layer, we have $s_n = n^2$ and the problem is that of summing the successive square numbers. Defining S_n to be the number of balls in the square pyramid with n layers, we can write the recurrence relation $S_n = S_{n-1} + s_n$ with $S_0 = 0$ and $n = 1,2,3,\ldots$.

As with the tetrahedral example, we can find an explicit formula for S_n depending only on n. The formula is

$$S_n = \frac{n}{6}(2n+1)(n+1)$$

and we call the sequence of numbers generated by this formula the *square pyramidal numbers*.

We leave it as an exercise for the reader to develop similar formulae for the numbers of balls in rectangle-based stacks.

Stacking

The first part of Sir Walter Raleigh's question, as we have given it, concerned determining the size of the base needed for a given number of cannon balls. Suppose a square pyramid is desired and there are m balls available. We can put

$$\begin{aligned} m &= S_n \\ &= \frac{n}{6}(2n+1)(n+1) \end{aligned}$$

and try to solve this equation for n, the number of layers, which is also the number of balls in the side of the square base. Expanded and rearranged, the equation is $\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}-m=0$.

Usually, the solution to this cubic equation, given a particular m, will not be a whole number and since there definitely is a whole number of cannon balls, the pyramid will be partially incomplete.

If m happens to be a square pyramidal number, then n will be an integer – the number of layers in the pyramid. Otherwise, m lies between two square pyramidal numbers and we would take the index of the larger of them to be the side of the required square. For example, without actually solving the equation (which is difficult), given 355 cannon balls we might begin to generate the sequence of square pyramidal numbers: $1, 5, 14, 30, 55, 91, 140, 204, 285, 385, \ldots$ and since $S_{10} = 385$ is the next pyramidal number larger than the number of balls, we would conclude that a square base of side $n = 10$ is required.

A truncated square pyramid

In the case of 355 cannon balls, it happens that another 30 balls are needed to complete the S_{10} stack. This, coincidentally, is a square pyramidal number, namely S_4. Therefore, 355 cannon balls forms a square pyramid missing the top four layers. In this sense, $355 = S_{10} - S_4$ is a *truncated square pyramidal number*. One might wonder what other numbers can be expressed as differences $S_n - S_k$ between square pyramidal numbers, and thus represent truncated square pyramids.

There are 44 distinct truncated square pyramidal numbers less than S_{10}. There would be 45 were it not for the fact that one number in this range, 25, can be expressed as a truncated pyramid in two ways: $S_4 - S_2$ and $S_5 - S_4$.

In general, given a square pyramid S_n, truncated pyramids can be formed by removing from the top any one of the pyramids $S_1, S_2, S_3, \ldots, S_{n-1}$. That is, there are $n-1$ non-trivial differences $S_n - S_k$ and thus $n-1$ truncated square pyramidal numbers that can be produced from the square pyramidal number S_n. Similarly, there are $n-2$ truncated square pyramidal numbers obtainable from the square pyramidal number S_{n-1}, and so on.

The sum of all these is $(n-1)+(n-2)+\cdots+1=\frac{n(n-1)}{2}$ but this number must be regarded as an upper bound on the number of distinct truncated square pyramidal cannon ball stacks that can be constructed from a collection of n balls as some of the stacks might arise in two or more different ways as was the case for the 25 stack.

We might wonder whether, having removed some number of layers from the top of a square rectangular pyramid, the resulting truncated pyramid could ever be reshaped into a complete pyramid. That is, can it happen that $S_m - S_n = S_k$?

A search through the square pyramidal numbers up to S_{539} revealed just eighteen such instances. They come in pairs since

$$S_m - S_n = S_k \Leftrightarrow S_m - S_k = S_n$$

and so there are just nine sums $S_m = S_n + S_k$ up to $m = 539$.

In looking for sums of pyramidal numbers that are themselves pyramidal numbers, we are seeking integer solutions to the equation

$$n(2n+1)(n+1) + k(2k+1)(k+1) = m(2m+1)(m+1)$$

This is analogous to looking for integer solutions to the Pythagorean equation

$$x^2 + y^2 = z^2$$

of which there are infinitely many. However, the square pyramidal numbers are represented as values of a cubic polynomial. It is interesting that two integer values of the cubic polynomial can sum to a third integer value of the polynomial. In contrast, it has long been known that

$$x^3 + y^3 = z^3$$

has no integer solutions.

Such investigations lead to questions in number theory. We go a short distance down this path in the section 'Some Diophantine equations'.

Squares

Every square number represents the number of cannon balls in the base of a square pyramid. In this sense, every square is a truncated square pyramidal number. Some, perhaps infinitely many, truncated square pyramidal numbers are also square numbers. For example,

$$S_4 - S_2 = 30 - 5 = 25 = 5^2$$

and

$$S_{21} - S_{19} = 3311 - 2470 = 841 = 29^2$$

and

$$S_{28} - S_{17} = 7714 - 1785 = 5929 = 77^2$$

In general, these are examples of integer solutions to the equation

$$\frac{n}{6}(2n+1)(n+1) - \frac{m}{6}(2m+1)(m+1) = k^2$$

We might also consider the somewhat related equation

$$\frac{n}{6}(2n+1)(n+1) = k^2$$

equating a single square pyramidal number to a square.

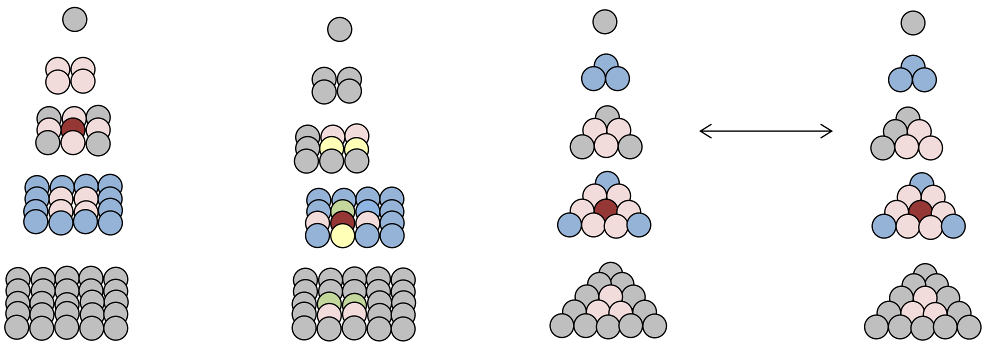
Many people have observed that $S_1 = 1 = 1^2$ and $S_{24} = 4900 = 70^2$ are, at once, square pyramidal numbers and squares. Edouard Lucas in 1875 conjectured that no other numbers have this property. Over 40 years were to elapse before G.N. Watson gave a proof of this in 1918. The proof involved advanced ideas in algebraic geometry. Thus, Sir Walter Raleigh's relatively simple musing about cannon balls has had far from simple mathematical consequences.

Sphere packing

An associated question concerns the average amount of filled space that occurs when equal spheres are packed in various ways. Harriot was interested in this aspect because he believed in the existence of atoms and therefore saw that sphere packing might relate to the fine structure of solid matter.

The phrase *packing density* means the proportion of space in a region inside the stack that is filled by spheres. The astronomer Johannes Kepler, in 1611, in the time of Thomas Harriot, conjectured that the sphere packing density that occurs in the cases of cannon balls and of fruit in a green grocer's shop, was the densest possible, whether the packing formed a regular pattern or not. In 1831, Carl Friedrich Gauss confirmed the conjecture in the case of regular packings of equal spheres but the final resolution of the issue did not occur until 2014 with a computer assisted proof by Thomas Hales.

Consider a ball inside a stack of balls. (For example, observe the red ball in each of the following diagrams. We imagine the layers of balls extending indefinitely.)



The red ball and hence every ball inside a stack is touching 12 neighbours. Comparison of the third and fourth stack arrangements shows that there are two possible orientations of the layers of balls above and below the layer containing the red ball. This makes no difference to the packing density but it is of interest to crystallographers.

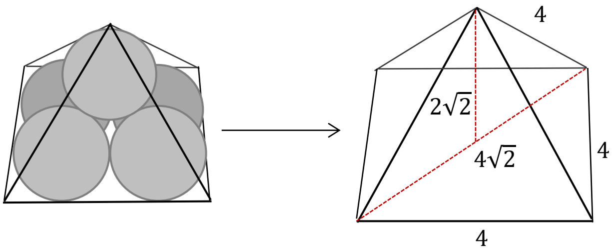
The first, second and third arrangements are the same from the crystallographic point of view. The colouring of the balls in the second stack is intended to show that this stack is the same as the third when an oblique plane through it (containing the pink balls) is considered.

Density

It is possible to think of a rectangular prism inside the stack and to count or calculate the number of balls it contains. The packing density is the ratio of the volume of the balls to the volume of the prism. In fact, it can be shown* that a box with dimensions $2 \times 2 \times \sqrt{2}$ inside the stack contains exactly 1 unit sphere. The box with these dimensions is sufficient to enclose a single sphere provided it is sliced in half first and then one of the halves is sliced again into four pieces. The pieces pack into the box in such a way that the box can join with similar adjacent boxes to make the repeating cannon ball pattern.

Then, since the volume of the sphere is $\frac{4\pi}{3}$, the ratio must be $\frac{\frac{4\pi}{3}}{4\sqrt{2}}$ which is approximately 0.74.

However, there can be more than one way to a result in mathematics. Imagine a square pyramid made by lines drawn tangent to the spheres forming the shape of the stack, as in the diagram below. We assume the spheres have radius 1 unit.



The volume of the five balls in the pyramid shown is $5 \times \frac{4\pi}{3}$ and the volume of the superimposed pyramid is $\frac{1}{3} \times 4^2 \times 2\sqrt{2}$. The ratio of these two volumes is greater than 1 because the sides of the stack are rough and the balls protrude a little outside the imaginary pyramid. Thus, this estimate of the packing density is inaccurate. However, it can be improved by doing the same thing with a larger pyramid. Its sides will be comparatively less rough and will correspond better with the outline.

A stack with base $n \times n$ has $\frac{1}{6}n(n+1)(2n+1)$ balls, and so a volume of $\frac{1}{6}n(n+1)(2n+1) \times \frac{4\pi}{3}$. The outline pyramid has base $2n \times 2n$ and height $2n\sqrt{2}$, so that its volume is $\frac{8n^3\sqrt{2}}{3}$. The ratio formed from these two volumes simplifies to $\frac{\pi(2n^2+3n+1)}{6n^2\sqrt{2}}$. So, for example, if $n = 10$, the ratio has reduced to about 0.86 because the roughness of the sides of the stack is having a smaller influence. We examine what happens in the limit if the base side n is made arbitrarily large and consequently, the sides become almost smooth.

By dividing through by the n^2 in the denominator, the ratio $\frac{\pi(2n^2+3n+1)}{6n^2\sqrt{2}}$ becomes

$$\frac{\pi}{6\sqrt{2}}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

When n is very large, the two fractions in the bracket become negligibly small and therefore the ratio approaches $\frac{\pi}{3\sqrt{2}} \approx 0.74$, as before.

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*A video accessible at https://www.youtube.com/watch?v=CROeIGfr3gs: The Best Way to Pack Spheres – Numberphile, demonstrates this.