Cakes and pizzas

This is a well-known dissection puzzle. We begin with a two-dimensional version and then consider its three-dimensional analogue.

- Suppose you are allowed to make four straight cuts across the surface of a pizza. What is the maximum number of pieces of pizza that can be obtained in this way?
  Also, what is the minimum number of pieces obtainable with four cuts? Is it possible to obtain each number of pieces between the minimum and the maximum with the four cuts?

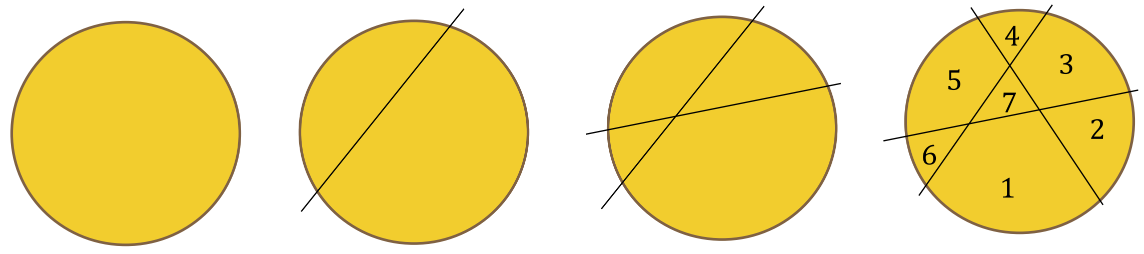
- The four faces of a tetrahedron can be imagined as belonging to planes extending into 3-space. If the tetrahedron is completely enclosed within a large sphere without touching it, the planes divide the sphere into separate regions. How many regions inside the sphere are created by the four planes?

We found the three-dimensional example in Ivan Moscovich's Mastermind Collection: Sensational Shape Problems & Other Puzzles.

Information about the three-dimensional version is available from the Online Encyclopedia of Integer Sequences, http://oeis.org/A000125, founded by N.J.A. Sloan; and the two dimensional case is discussed at http://oeis.org/A000124.

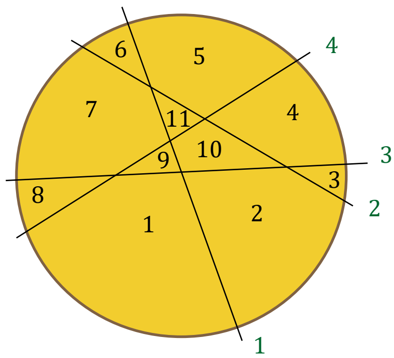
Pizzas

In the statement of the problem, four cuts across the pizza were specified. However, it would be natural to start with fewer cuts and increase the number progressively to see how many regions can be produced at each stage. An even more interesting question would be to find the maximum number of pieces that can be obtained with some unspecified number n of cuts. The general solution, if it exists, would be in the form of a relation between the number of lines n and the number of regions r produced by the lines.

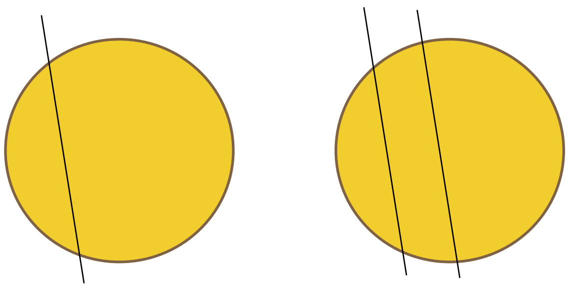


Starting from $n = (0,1,2,3,\ldots)$ we might form a sequence of the corresponding numbers of regions. By experiment, its first few terms appear to be $r = (1,2,4,7,\ldots)$. We might wonder whether the fourth term, when $n = 3$, should really be $r = 8$ and we have merely failed to find the right way to make the cuts. After all, each point of the plane is on one or other of two sides of each line, making $2 \times 2 \times 2 = 8$ alternatives. Further mathematical thinking is needed to confirm that 7 is indeed correct and that not all of the 8 combinations are feasible.

With $n = 4$, we might construct a drawing like the following and be tempted to conclude that the solution to the puzzle is $r = 11$. But, can we be sure? Not finding how to get a bigger number for r does not guarantee that one does not exist. Again, there should be a logical explanation.

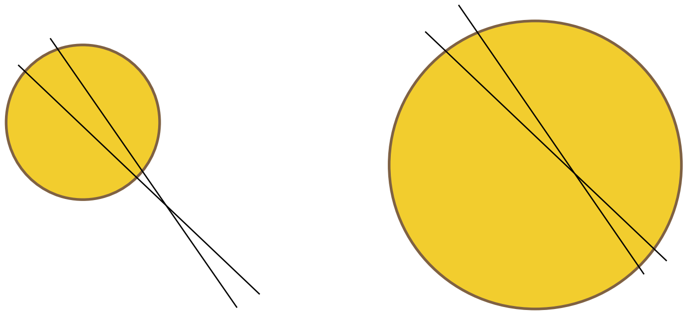


Turning our attention to the minimum numbers of pieces, we can argue that adding a line parallel to any existing lines always adds at least one region.



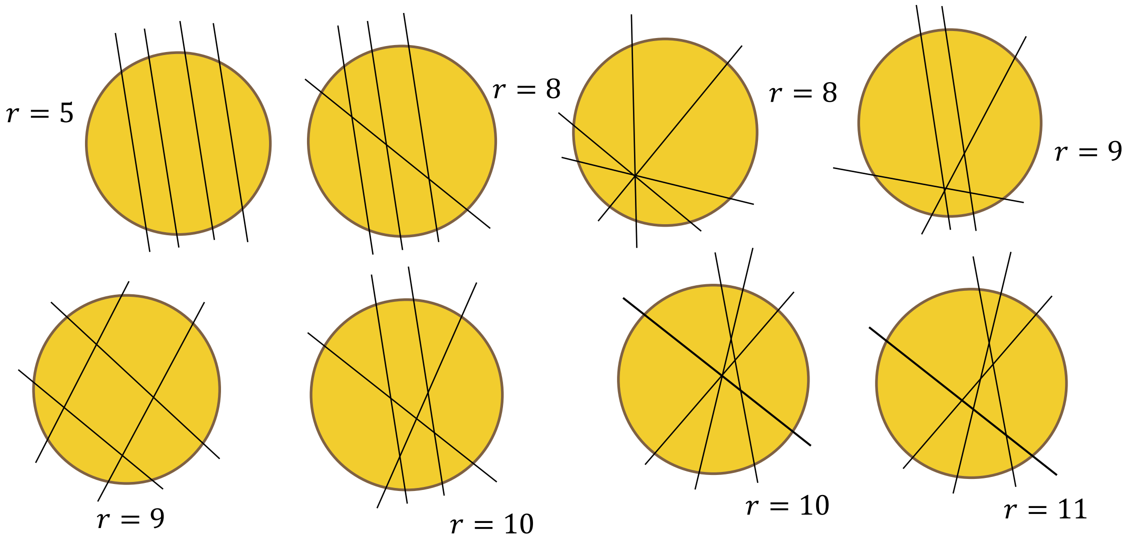
With no lines there is one region, with one line there are two regions, with two lines there are at least three regions, and so on, so that the minimum number of regions must always be one greater than the number of lines. We write, $N_{\min}(r) = n + 1$.

The key to producing the minimum number of regions seems to be to make the lines parallel. We *could* get the minimum number of regions with two non-parallel lines in a pizza of limited size but the same lines do not produce this minimum if the pizza is made large enough.



So, we will insist on parallel lines when seeking the minimum number of regions and we will allow the pizza to be large enough to include all the intersections of any non-parallel cuts.

With four cuts, we can find arrangements like the following given the rule about parallel lines.



Thus, unless there are others that we have not considered, there are no arrangements of four lines that produce 6 or 7 regions. An argument that 6 or 7 regions are not possible runs as follows.

The minimum number $n + 1$ of regions occurs when n lines are parallel. If one line is temporarily removed, there are $n - 1$ lines and therefore n regions. When the nth line is reinserted so that it is not parallel with the others, it crosses the $n - 1$ lines and therefore creates n new regions making $2n$ regions in total. Therefore, no number of regions between $n + 1$ and $2n$ is possible.

In the example, $n = 4$, so that $N_{\min}(r) = 5$ and $2n = 8$. Hence, we are confident that $r = 6$ and $r = 7$ are impossible when $n = 4$.

The experiment of arranging four cuts across a pizza suggests that the number of regions is less than the maximum whenever more than two lines intersect at a single point. Compare the last two drawings above. They are identical except for the position of one line. It is clear that if three lines intersect at a point, a parallel motion of one of the lines away from that point creates an extra region.

Therefore, in seeking an arrangement of cuts that produces the most regions, we must ensure that no two cuts are parallel and that each intersection involves only two of them.

The 1-*dimensional case*
When points are marked on a line, they divide the line into 1-dimensional regions. With no points marked there is one region, with one point there are two regions, and with every additional point marked there is a new region created. We have the relation

$$r_{n+1} = r_n + 1, \qquad n = 0,1,2,\ldots, \quad r_0 = 1$$

where r_n is the number of regions produced by n points on a line.

When this is written out term-by-term, we have the sequence $(1,2,3,\ldots)$. That is,

$$r_n = n + 1, \quad n = 0,1,2,\ldots$$

The 1-dimensional case is rather obvious but we will use it to confirm what we suspect is true in two dimensions.

The 2-*dimensional case*
Suppose there are k lines in the plane, no pair of them parallel and no more than two of them ever passing through a common point. Assume for the moment that the number of regions produced by the k lines is maximal.

A new line is added with the same rules about parallelism and intersections. It makes a new region as it passes through each existing region since the region is cut into two parts. To count the number of existing regions that the line passes through, and hence the number of new regions produced, observe that since it is not parallel to any of them the new line intersects each of the existing lines.

Thus, the new line acquires k points of intersection that, according to the 1-dimensional case, divide it into $k+1$ regions. These $k+1$ regions on the line correspond to the 2-dimensional regions that the line passes through. Hence, the $(k+1)$th line creates $k+1$ new 2-dimensional regions. Moreover, the number of new regions must be maximal for $k+1$ lines because the new line passes through the greatest possible number of existing regions. If there were r_k regions when there were k lines, it follows that the number of regions when there are $k+1$ lines is $r_{k+1} = r_k + k + 1$.

This relation clearly holds for $k = 0$ and for $k = 1$, and so, by the argument in the previous paragraph, it must hold for $r_3, r_4, r_5, \ldots$ and so on. We conclude that the maximal number of regions created by n lines in the plane is given recursively by

$$r_{n+1} = r_n + n + 1, \;\; n = (0,1,2,\ldots), \;\; r_0 = 1$$

The first few terms of the sequence r_n are therefore $(1,2,4,7,11,16,\ldots)$.

To generate the nth term of this sequence without calculating all the previous terms, we need an explicit formula, one that expresses r_n as a function only of n. We give two methods for finding such a function in another chapter. For now, we state that the required formula is

$$r_n = \frac{n(n+1)}{2} + 1$$

As an aside, we note that the expression $\frac{n(n+1)}{2}$ gives the nth *triangular* number t_n so that

$$r_n = t_n + 1$$

The 3-*dimensional case*

Planes divide 3-dimensional space in ways that are quite hard to visualise. In the book by Ivan Moscovich, the solution is given with the help of a diagram. It is claimed that the four planes enclosing a tetrahedron create a region at each vertex, a region at each edge, a region at each face and the region inside the tetrahedron. Thus, there are $4 + 6 + 4 + 1 = 15$ regions.

We would like to be able to see this in a systematic way that can be generalised for any number of planes.

In a manner similar to the 2-dimensional case, we can argue that the number of regions produced by a collection of intersecting planes is maximised when no two planes are parallel, when no two lines of intersection are parallel, and when no more than three planes intersect at any point.

Suppose there are already k planes arranged to make the maximum number c_k of 3-dimensional regions. When another plane is inserted, again complying with the rules about parallelism and intersections, it intersects every one of the existing k planes. As a result, k lines of intersection appear on the new plane. According to the 2-dimensional case, these k lines of intersection define r_k regions on the new plane. We see that each of these regions in the plane is a boundary created within a 3-dimensional region, dividing it into two parts. Thus, the $(k + 1)$th plane causes the creation of r_k new 3-dimensional regions. We have $c_{k+1} = c_k + r_k$.

A table illustrates this process.

n	1	2	3	4
r_n	2	4	7	11
c_n	2	4	8	15

(The letter c has been used for the numbers of 3-dimensional regions because they are known as *cake* numbers.)

As before, we would like to find an explicit formula for c_n instead of the recurrence relation $c_{n+1} = c_n + r_n$. The details of how this can be done are in another chapter. We give the following function definition as the solution to the recurrence relation for the cake numbers.

$$c_n = \frac{n(n^2 + 5)}{6} + 1$$