Some recurrence relations

In the section on pizzas and cakes, we saw the recursive formula $r_{n+1} = r_n + n + 1$ from which the number of pieces created by n cuts across a pizza can be calculated (although somewhat laboriously). Similarly, in 3-dimensions we found the recurrence relation $c_{n+1} = c_n + r_n$ for the so-called *cake numbers*.

In these and similar situations we look for more efficient formulas that give the same results but with a fixed small number of calculation steps. Thus, for example, we claim that the maximal number of pieces into which a pizza can be cut with n cuts is given by $r_n = \frac{n(n+1)}{2} + 1$, which gives r as a function of n only and which can be computed in four steps whatever the size of n.

We explore two possible approaches to finding an explicit function from a recurrence relation.

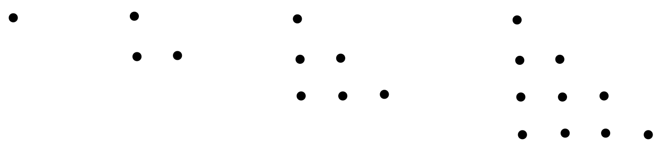
1. Rates of change

There is information about a rate of change implicit in a recurrence relation.

A recurrence relation shows the *difference* between successive terms of a sequence. For example, the recurrence relation $r_{n+1} = r_n + n + 1$ shows that the difference in the values for r as n changes by 1 unit is just $r_{n+1} - r_n = n + 1$. In this case, the difference is linearly related to the number n.

A connection can be made with polynomial functions. It is easily checked that for any second-degree polynomial function $f(n) = an^2 + bn + c$, , the difference in the values for f as n changes by 1 unit is $f(n+1) - f(n) = 2an + a + b$, which is again a linear function of n. Therefore, when a recurrence relation expresses a linear difference, we have a reason to expect that its explicit form will be a quadratic function.

Triangular numbers
The triangular numbers provide an example. These are the numbers forming the sequence $(1,3,6,10,\ldots)$ in which the difference between successive terms increases by 1 at each step.



The recurrence relation that describes them is

$$t_{n+1} = t_n + n + 1, \qquad n = 1,2,3,\ldots, \qquad t_1 = 1$$

The difference $t_{n+1} - t_n = n + 1$ is a linear function of n, suggesting that the required explicit function will be quadratic. For a general quadratic function given by $f(n) = an^2 + bn + c$, we see that in order to get $f(n+1) - f(n) = 2an + a + b = n + 1$, we must have $2a = 1$ and $a + b = 1$. Hence, $a = \frac{1}{2}$ and $b = \frac{1}{2}$. Thus, $f(n) = \frac{n^2}{2} + \frac{n}{2} + c$.

Since the initial condition, from the first triangular number, is $f(1) = 1$, we set $c = 0$ and thus, we have constructed a function, defined on the integers, given by $f(n) = \frac{n^2+n}{2}$.

We can write this as

$$t_n = \frac{n(n+1)}{2}$$

It remains to check that the formula satisfies the recurrence relation for all values of n. We see that $t_{n+1} = \frac{(n+1)(n+2)}{2}$ and therefore, $t_{n+1} - t_n = (n+1)\left(\frac{n+2}{2} - \frac{n}{2}\right) = n + 1$, as required.

Lines dividing a plane

The maximal number of regions created by n lines crossing a plane is given by the recurrence relation $r_{n+1} = r_n + n + 1$. This is almost the same recurrence relation as the one expressing the triangular numbers. It is not exactly the same because it has a different starting number.

As before, in order to have $f(n+1) - f(n) = n + 1$, we must have $2a = 1$ and $a + b = 1$. Hence, $a = \frac{1}{2}$ and $b = \frac{1}{2}$. Thus, $f(n) = \frac{n^2}{2} + \frac{n}{2} + c$. However, in this case we require $f(0) = 1$. So, we deduce $c = 1$ and therefore $f(n) = \frac{n^2}{2} + \frac{n}{2} + 1$ or

$$r_n = \frac{n(n+1)}{2} + 1$$

We can accept this as the required explicit formula after checking that it gives the correct first term and that it satisfies the recurrence relation.

Planes cutting 3-space

From the recurrence relation $c_{n+1} = c_n + r_n$ we see that $c_{n+1} - c_n = r_n$ and from the previous result, it is clear that the differences between cake numbers follow a quadratic rule. We conjecture that the explicit function we seek for c_n is a cubic polynomial.

The general cubic polynomial function can be expressed as $g(n) = an^3 + bn^2 + cn + d$. The difference $g(n+1) - g(n)$ turns out to be $3an^2 + (3a + 2b)n + a + b + c$. In comparison, the recurrence relation gives $c_{n+1} - c_n = r_n = \frac{1}{2}n^2 + \frac{1}{2}n + 1$.

For these polynomial expressions to be equal, the corresponding coefficients must be equal. Therefore, $3a = \frac{1}{2}$, $3a + 2b = \frac{1}{2}$ and $a + b + c = 1$. Hence, $a = \frac{1}{6}$, $b = 0$ and $c = \frac{5}{6}$.

Thus, the cubic function we seek is $g(n) = \frac{1}{6}n^3 + \frac{5}{6}n + d$. Since $g(0) = 1$, we put $d = 1$ so that $g(n) = \frac{n(n^2+5)}{6} + 1$ and after checking that this function does satisfy the recurrence relation, we conclude that the cake numbers are given by $c_n = \frac{n(n^2+5)}{6} + 1$, as claimed.

Square pyramidal numbers

Spheres can be stacked in layers with each layer in the form of a square. In this way a square pyramid shape is built. Depending on the number of layers, the numbers of spheres in each pyramid make a sequence of square pyramid numbers $(1, 5, 14, 30, \dots)$.

The recurrence relation for successive square pyramidal numbers is $S_{n+1} = S_n + (n+1)^2$. As with the cake numbers, we see that the difference $S_{n+1} - S_n$ is a quadratic function of n and so we expect the explicit function S_n to be given by a cubic polynomial.

Following the same steps as for the cake numbers, we arrive at the explicit formula

$$S_n = \frac{n(n+1)(2n+1)}{6}$$

and check that this gives $S_1 = 1$ and that the recurrence relation is satisfied for all n.

2. linear algebra

As mentioned, the differences between values of a quadratic at successive integers grow linearly. Therefore, in considering a recurrence relation like $r_{n+1} = r_n + n + 1$ in which the differences $r_{n+1} - r_n$ grow as a linear function of n, it is reasonable to expect that the explicit solution to the recurrence relation is a quadratic function.

We can use the known first few values of the sequence generated by the recurrence relation to obtain a system of simultaneous equations, with a view to determining the coefficients in the quadratic. Thus, assuming $r_n = an^2 + bn + c$, we have in the case of the pizza cuts puzzle

$$\begin{aligned} 2 &= a + b + c \\ 4 &= 4a + 2b + c \\ 7 &= 9a + 3b + c \end{aligned}$$

The system has the unique solution $a = \frac{1}{2}, b = \frac{1}{2}, c = 1$ leading to the explicit function $r_n = \frac{1}{2}n^2 + \frac{1}{2}n + 1$ which can be written

$$r_n = \frac{n(n+1)}{2} + 1$$

To confirm that this is the solution, it is readily checked that

$$r_{n+1} - r_n = n + 1$$

as required, and the initial value $r_1 = 2$ is correctly given.

In a similar way, we assume that the cake numbers c_n have the form $c_n = an^3 + bn^2 + cn + d$. From the first four known values we have the system

$$\begin{aligned} 2 &= a + b + c + d \\ 4 &= 8a + 4b + 2c + d \\ 8 &= 27a + 9b + 3c + d \\ 15 &= 64a + 16b + 4c + d \end{aligned}$$

which has the solution $a = \frac{1}{6}, b = 0, c = \frac{5}{6}, d = 1$. Hence, we propose $c_n = \frac{1}{6}n^3 + \frac{5}{6}n + 1$ and after checking that this function satisfies the recurrence relation, we conclude, as before, that

$$c_n = \frac{n(n^2+5)}{6} + 1$$