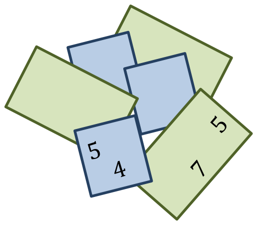
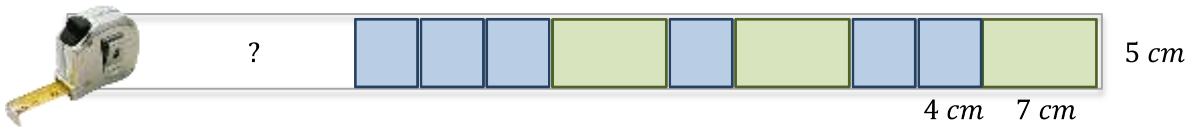
A puzzle with tiles





You have a plentiful supply of ceramic tiles that come in two sizes, 5×4 cm^2 and 5×7 cm^2. The tiles are needed to cover a strip 5 cm high and 902 cm long. No tiles are to be cut and the strip is to be covered exactly. Any number of tiles of either size may be used.

- Can this be done?
- Assuming the tiling is possible, what is the smallest number of tiles required?
- What combination would use the most tiles to achieve the tiling?

Puzzles similar to this one are often used to motivate the study of ideas about numbers and divisibility going back to the time of Euclid. However, a straightforward approach is possible, avoiding complications of the Ancient Greek kind. Before long, we may find that some deeper mathematical ideas are needed but we can begin intuitively and leave the more standard, general, rigorous and difficult version until the article on linear Diophantine equations.

If the strip could be covered using just the tiles with the 7 cm edge, we would have used fewest tiles, but this is not possible because 7 does not divide 902 exactly. If the tiling could be done using only the 4 cm tiles, the largest number of tiles would have been used, but again this is impossible because 4 does not divide 902. Evidently, some combination of the two types of tiles is needed.

A line of 129 of the 7 cm tiles is 1 cm too long. However, if 3 of these tiles are removed and replaced by 5 of the 4 cm tiles, the strip has exactly the correct length. That is,

$$126 \times 7 + 5 \times 4 = 902.$$

In this solution, 131 tiles have been used.

At the other extreme, a line of 226 of the 4 cm tiles is too long by 2 cm. The line is shortened by 2 cm if 4 of these tiles are removed and replaced by 2 of the 7 cm tiles. That is,

$$2 \times 7 + 222 \times 4 = 902.$$

This solution uses 224 tiles.

We suspect that other solutions exist that involve intermediate numbers of tiles. For example, starting from the first solution, we might remove 4 of the 7 cm tiles and add 7 of the 4 cm tiles. Since exactly 28 cm has been added and subtracted from the length, another solution has been found. Namely, using 134 tiles,

$$122 \times 7 + 12 \times 4 = 902.$$

This solution uses three tiles more than the first solution and there is nothing to prevent the same exchange of 4 lots of 7 for 7 lots of 4 being performed repeatedly to obtain many more solutions. Thus, the number of tiles used at each step would be three more than in the previous solution, so that the minimum, 131, grows to the maximum, 224, in 31 steps. This means there must be 32 distinct solutions, except for different arrangements of the tiles.

Looking at the range of solutions, we observe that as the number of 7 cm tiles decreases from 126 to 2, the number of 4 cm tiles increases from 5 to 222. We might wonder whether there is a solution with the same number of the two sizes of tiles. That is, is there a number n such that $7n + 4n = 902$? Clearly, there is.

$$82 \times 7 + 82 \times 4 = 902.$$

The numbers in this puzzle appear to have been chosen so that everything works out cleanly. What if, instead, the tiles had been of sizes 8×5 and 5×4? We could try using 113 of the larger tile, but this would make a 904 cm strip. There seems to be no way to replace some of the 8 cm tiles with 4 cm tiles so that the length is reduced to 902 cm. Similarly, a line of 226 of the smaller tiles is 2 cm too long and likewise cannot be fixed by swapping some numbers of larger and smaller tiles.

In fact, any combination $4x + 8y$ where x and y are integers must be a number divisible by 4 (since $4x + 8y = 4(x + 2y)$) but 902 is not exactly divisible by 4 and therefore it is impossible to obtain that length with these tile sizes.

The tile puzzle is an instance of a more general class of problems. We might replace the fixed tile lengths by coefficients a and b, assumed to be non-zero, and the numbers of tiles by variables x and y; and then write $ax + by = k$ where k is a constant.

When x and y are required to be integers, the equation $ax + by = k$ is called a Diophantine equation. We deal with this particular Diophantine equation more thoroughly in the article *A Diophantine equation*.