TIP 14 and 15: *Gardner Triples Supplement*

This piece follows up on two previous TIPs about Gardner Triples (*GT's*). *GTs* arose from a geometric diagram that led to the recurrence $\left(a, b, \frac{ab+t^2}{b-a}\right)$ in which the third term is an integer. Equivalently, we took $a, m, n, t \in Z^+$, $m \leq n$, with a and t coprime, and formed the general *GT*, denoted

$$(a, a+m, a+n)_t$$

where $mn = S = a^2 + t^2$. The triple describes the identity $\tan^{-1}\frac{t}{a} = \tan^{-1}\frac{t}{a+m} + \tan^{-1}\frac{t}{a+n}$.
When $t = 1$ (the type 1 Gardner Triple) the subscript is dropped.

For example, for $a = 3$ and $t = 4$ we have $S = 25$ and thus can choose $(m, n) = (1{,}25)$ so that the *GT* $(3{,}4{,}28)_4$ describes the identity $\tan^{-1}\frac{4}{3} = \tan^{-1} 1 + \tan^{-1}\frac{1}{7}$.

Linking Gardner Triples

Sometimes it is possible to link *GTs* together. For example, the *GT* with $a = 1$ and $t = 1$ becomes $(1{,}2{,}3)$ and likewise the triple with $a = 3$ and $t = 1$ becomes $(3{,}5{,}8)$.

Thus, because $\tan^{-1} 1 = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$, and $\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$, we can write a combined statement $\tan^{-1} 1 = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$ and denote it as a 4-tuple $(1{,}2{,}5{,}8)$.

Again, since $(8{,}13{,}21)$ is also a *GT*, we can develop the 5-tuple $(1{,}2{,}5{,}13{,}21)$. This process ultimately develops a beautiful statement linking π to the odd numbered Fibonacci terms

$$\tan^{-1} 1 = \frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{13} + \tan^{-1}\frac{1}{34} + \tan^{-1}\frac{1}{89} + \cdots.$$

That is to say, $\pi = 4\sum_{n=1}^{\infty}\left(\frac{1}{F_{2n+1}}\right)$, where F_n is the nth Fibonacci number.

More generally, by considering any type t triple $(a, a+m, a+n)_t$, it is possible to create a new triple whose first term is any term of this triple, thereby enabling an infinite linking process.
To see how this is done with the third term as the link, we might relabel the general *GT* as $(a_1, a_1 + m_1, a_1 + n_1)_t$ and think about a new $a_2 = a_1 + n_1$.

We form a new sum of squares, $S_2 = {a_2}^2 + t^2 = (a_1 + n_1)^2 + t^2$. This new sum simplifies to $S_2 = S_1 + n_1(2a_1 + n_1) = n_1(m_1 + n_1 + 2a_1)$. In this form it becomes clear that we can choose $m_2 = n_1$ and $n_2 = (m_1 + n_1 + 2a_1)$ to form a second generation triple of the form $[a_2, a_2 + n, a_2 + (m_1 + n_1 + 2a_1)]_t$. From this linking triple, we can build a third, fourth or fifth generation triple, and so on, all of which are related to the very first *GT*.

For example, the *GT* $(3{,}4{,}28)_4$ mentioned above, with $(m_1, n_1) = (1{,}25)$, combines with $(28{,}53{,}60)_4$ to produce the 4-tuple $(3, 4, 53, 60)_4$.

We can easily check that $\tan^{-1}\frac{4}{3} = \tan^{-1} 1 + \tan^{-1}\frac{4}{53} + \tan^{-1}\frac{1}{15}$.

Using the same strategy, with $a_2 = 28$, $m_2 = 25$ and $n_2 = 32$ we can link $(28{,}53{,}60)_4$ to the triple $(60{,}92{,}173)_4$ and produce the 5-tuple $(3, 4, 53, 92, 173\,)_4$. If we continue in this manner, we develop the n-tuple $(3, 4, 53, 92, 286, 703, 1886, \dots)_4$. In a similar way, any term of a tuple can be taken to be the third term of some *GT* and thus, can be replaced by a pair of terms.

We can generate any number of expressions for π using this strategy. For example, with $a = 2$ and $t = 2$, perhaps using a spreadsheet program, we can generate the linked *GT*s $(2, 3, 10)_2$, $(10, 18, 23)_2$, $(23, 36, 64)_2$, $(64, 105, 164)_2$, $(164, 264, 433)_2$, …, making the n-tuple $(2, 3, 18, 36, 105, 264, \dots)_2$. From this we deduce

$$\pi = 4\left[\tan^{-1}\frac{2}{3} + \tan^{-1}\frac{1}{9} + \tan^{-1}\frac{1}{18} + \tan^{-1}\frac{2}{105} + \tan^{-1}\frac{1}{132} + \cdots\right]$$

If we relax the definitions slightly, allowing $t = \sqrt{3}$, then we can show that

$$\pi = 3\left[\tan^{-1}\frac{\sqrt{3}}{2} + \tan^{-1}\frac{\sqrt{3}}{9} + \tan^{-1}\frac{\sqrt{3}}{19} + \tan^{-1}\frac{\sqrt{3}}{54} + \tan^{-1}\frac{\sqrt{3}}{137} + \tan^{-1}\frac{\sqrt{3}}{363} + \cdots\right]$$

Machin's Formula from Gardner Triples

John Machin (1686 -1751) a professor of astronomy at Gresham College London, in 1706 developed a quickly converging series for π and used it to calculate the number correct to 100 decimal places. Contrast that with the efforts of Ludolph van Ceulen (1540 - 1610) the German mathematician who spent twenty five years of his life working out just 35 decimal places.

Machin's formula states that $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$. We reverse engineered this result to show that it can be derived using Gardner triples.

If $a = 119$ and $t = 120$, then $S = 2856 = 191^2$. We can choose $m = n = 191$, giving the *GT* $(119, 288, 288)_{120}$. This amounts to saying that $\tan^{-1}\frac{120}{119} = 2\tan^{-1}\frac{120}{288}$. (As an aside, we note that this is an interesting choice because the set of numbers $119, 120, 191$ is a Pythagorean Triple and such instances give rise to special *GTs* where angles with rational tangents have half angles whose tangents remain rational.)

Then, putting $m = 1$ and $n = 120$, generates a second *GT* as $(119, 120, 28680)_{120}$.

This can be rewritten as $[(288, 288), 120, 28680]_{120}$ where the first element 119 has been replaced by a nested expression that represents the sum $2\tan^{-1}\frac{120}{288}$. The expression in square brackets is equivalent to $[(36, 36), 15, 3585]_{15}$ when the common factor 8 is eliminated from each of the four terms.

Thus, we have established that $2\tan^{-1}\frac{15}{36} = \tan^{-1} 1 + \tan^{-1}\frac{1}{239}$, which can be easily verified.

Using a similar strategy choose $a = 12$ and $t = 5$ so that $S = 169$. With $m = n = 13$, create the triple $(12, 25, 25)_5$ and scale it up to $(36,75,75)_{15}$ by introducing a common factor of 3.

The GT $[(36, 36), 15, 3585]_{15}$ can now be rewritten as $[(75, 75, 75, 75), 15, 3585]_{15}$ and by eliminating the common factor of 15 this new expression is reduced to $[(5, 5, 5, 5), 1, 239]_1$.

Hence $4\tan^{-1}\frac{1}{5} = \tan^{-1} 1 + \tan^{-1}\frac{1}{239}$ or, when simplified and rearranged,

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$