That integration can be done by antidifferentiation was a remarkable discovery by Newton and Leibniz, and it can be a thing of wonder for today's students. But how often do we neglect to tell them the awful truth that hardly any functions have an antiderivative expressible in terms of ordinary functions, and in most real cases another method must be used when integration is required.

What are often mis-called *integration techniques* should really be labelled *techniques for finding an antiderivative*. These are challenging, no doubt, in the same way as crossword puzzles are challenging. But, having a good vocabulary for crosswords does not make one a writer; and neither does knowledge of some antiderivatives make one a mathematician. A more mathematical goal would be to learn to identify when integration by any means is appropriate. Nevertheless, school curricula continue to prescribe *integration techniques*.

Consider a function $f(x) = \frac{1}{[g(x)]^n}$ where $n \in \mathbb{Z}^+$. Its derivative is $f'(x) = -\frac{ng\prime(x)}{[g(x)]^{n+1}}$.
The derivative of the function in the denominator appears in the numerator, so that in the unlikely event that an antiderivative $\int \frac{-ng\prime(x)}{[g(x)]^{n+1}} dx$ is needed, the solution is immediately available. The trick lies in recognising a derivative.

What may happen though is that the instructor will recognise an opportunity for a *substitution*, particularly when instead of only the derivative appearing in the numerator, a linear function of it occurs. That is, $\int \frac{Ag\prime(x)+B}{[g(x)]^{n+1}} dx$.

The recipe for this will be something like, *put* $u = g(x)$, *then* $x = g^{-1}(u)$, *and calculate* $\frac{du}{dx}$.
Mysteriously, the du and the dx are split up so that the dx can be replaced by something involving du, (notwithstanding an earlier insistence that $\frac{du}{dx}$ is one number, the limit of the ratio $\frac{\delta u}{\delta x}$ as $\delta x \to 0$).

Clearly, this recipe will be unpalatable for any student expecting mathematics to make sense. Rather better is replacing dx with $\frac{dx}{du} du$ but even so, a justification is needed dependent on understanding the differentiation of a function of a function. An explanation could go as follows, but this still needs prior work before it can be properly understood and believed.

Suppose u is a function of x that has an antiderivative and let $\int f(u)du = F(u)$. We wish to show that $\int f(u)u'dx$ is the same as $F(u)$. Clearly, differentiating $F(u)$ with respect to u gives $F'(u) = f(u)$, while by differentiating $F(u)$ with respect to x, we have $F'(u) \cdot u'$, or equivalently, $f(u) \cdot u'$. Then, reversing both differentiations we have on the one hand $F(u)$ and on the other, $\int f(u)u'dx$. Thus, the two are the same.

Challenge 8: Show rigorously that the function-of-a-function or chain rule procedure in differentiation is valid.