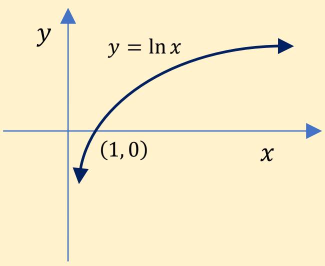
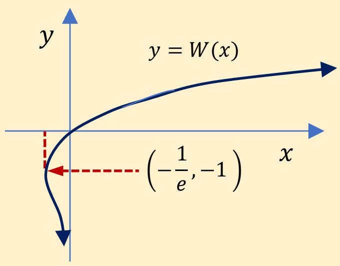
Sometimes in mathematics, while working to find a solution to an equation, an impasse is encountered that seemingly cannot be breached by using established methods. There is, for example, a rich mathematical history associated with overcoming algebraic obstacles in finding solutions to the general cubic, quartic and higher degree polynomial equations.

A particular equation, now rigorously defined, and which arises from many models in the natural sciences, had proved a stumbling block in the 18th century when it was first considered by Johann Lambert in 1758. What follows is a brief introduction to the concept.

We define $\ln\ x$ as the 'power' that the base e needs to be raised to, to equal x. The relationship between x and $\ln\ x$ is expressed in the identity $e^{\ln x} = x$ and, of course, tables (or a suitably programmed electronic device) are required to ascertain the quantity $\ln x$.

In a similar way, the Lambert W function $W(x)$ (also known as the product-log function) is defined as that power of e which, when multiplied by itself, produces x. The relationship is expressed in the equation $W(x) \cdot e^{W(x)} = x$, and electronic devices are similarly required to find the solution.





Note: When x is real, $W(x)$ is defined for $x \geq -e^{-1}$, and has two values when $-e^{-1} < x < 0$, and one value otherwise. To avoid confusion, we define the principal branch of $W(x)$, labelled $W_0(x)$ as that part of the curve where $W(x) \geq -1$.

If, at some line of algebraic simplification, a mathematician encounters an equation of the form $g(x) \cdot e^{g(x)} = h(x)$ then, on the next line, they can pronounce that $g(x) = W[h(x)]$.
In particular if $h(x)$ is the constant c, then $g(x) = W[c]$ and the right-hand side can be evaluated using tables or a suitable electronic device.

Example 1: Suppose we wish to know where the logarithmic curve $y = \ln x$ intersects the hyperbola $y = \frac{3}{x}$. On solving for x in the equation $\ln x = \frac{3}{x}$ the second line of working would be $x \ln x = 3$. Knowing that $x = e^{\ln x}$ we can then write $\ln x \cdot e^{\ln x} = 3$. It is now in the correct form, so we proceed by writing $\ln x = W(3)$ and finally $x = e^{W(3)}$. From the program Wolfram Alpha, we find $W(3) \approx 1.04991$ and thus $x = e^{1.04991} \approx 2.8574$.

Example 2: To solve $2^x = x^2$, $x < 0$, first write $x \ln 2 = 2 \ln|x|$ so that $x^{-1} \ln|x| = \frac{\ln 2}{2}$. Bearing in mind that $x < 0$, we write $\ln|x| = \ln(-x)$, and since $-x^{-1} = (-x)^{-1}$, we can write the equation as $(-x)^{-1} \ln(-x) = -\frac{\ln 2}{2}$, and then, $-\ln(-x)\, e^{-\ln(-x)} = \frac{\ln 2}{2}$. Thus, $-\ln(-x) = W\left(\ln\sqrt{2}\right)$ and the irrational negative solution $x \cong -0.7666646$ follows (approximately equal to $-\frac{23}{30}$).

Challenge 17: Find the three solutions to $2^x = x^2$ for all real x.