The American author Samuel Clemens popularised the saying "There are three types of lies: lies, damned lies and statistics". Simpson's paradox is certainly an instance where the adage rings true. It occurs when two or more independent sets of results each suggest a particular conclusion but, when combined, suggest exactly the opposite conclusion.

A striking example taken from Wikipedia details the success rates of extracting small and large kidney stones using two types of surgical methods, say method A and method B as shown in the table. The red numbers in brackets indicate the sample sizes in each of the four cases.

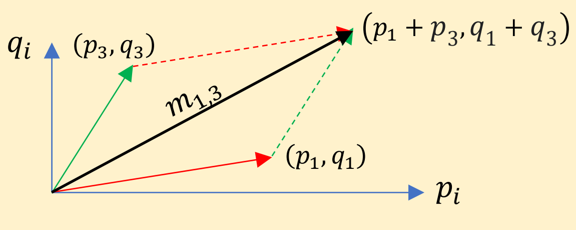
Kidney Stone Surgery	Method A (350 patients)	Method B (350 patients)
Small Stones	81 successes (87) 93%	234 successes (270) 87%
Large Stones	192 successes (263) 73%	55 successes (80) 69%

The results for both small and large kidney stone surgeries appear to confirm that Method A is the better surgical method with a 93% success rate for small stones and 73% success rate for large stones. We could imagine a hospital CEO feeling doubly confident that, because there are two independent results that confirm the efficacy of Method A, then Method A should be adopted for all future surgeries at the hospital.

However, if we look at the combined results for all surgeries, we find, for Method A, 273 successes out of a total of 350 patients for a success rate of 78%, and, for Method B, 289 successes out of 350 patients for a higher success rate of 83%. The previous conclusion is contradicted. How can this be?

The explanation for Simpson's paradox lies in considering the different sample sizes and the unorthodox way the fractions are being added. Generally, if the proportions of successes in four groups are $\frac{p_i}{q_i}, i = 1,2,3,4$, with $\frac{p_1}{q_1} > \frac{p_2}{q_2}$ and $\frac{p_3}{q_3} > \frac{p_4}{q_4}$,then it is not necessarily true that $\frac{p_1+p_3}{q_1+q_3} > \frac{p_2+p_4}{q_2+q_4}$.

In the kidney stone example, the odd subscripts refer to Method A and the even subscripts refer to Method B. The quantities $m_{1,3} = \frac{p_1+p_3}{q_1+q_3}$ and $m_{2,4} = \frac{p_2+p_4}{q_2+q_4}$ are called the mediants of their component fractions. We can illustrate the fractions and the mediant $m_{1,3}$, for example, as a vector with components on the p_i and q_i axes.



The gradient of the mediant vector is a measure of the mediant's size and is determined by the relative gradients of its component fractions. Accordingly, the paradox will occur whenever the gradient of the mediant $m_{1,3}$ exceeds the gradient of the mediant $m_{2,4}$.

Challenge 18: When will the mediant of $\frac{a}{b}$ and $\frac{c}{d}$ be equal to the mean of the two fractions?