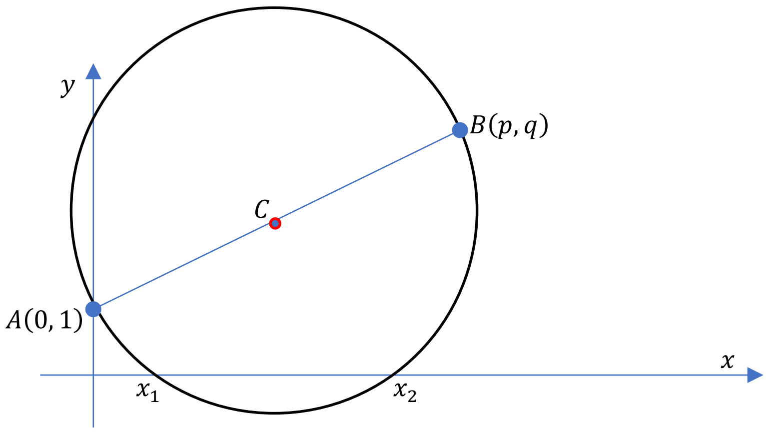
Thomas Carlyle (1795-1881), the British historian and mathematician, devised an ingenious geometric method for locating the roots of the quadratic equation $x^2 - px + q = 0$ involving a ruler, a compass and a sharp pencil. The method is described as follows.

Plot the points $A(0,1)$, $B(p,q)$ and their midpoint $C\left(\frac{p}{2}, \frac{1+q}{2}\right)$ on the cartesian plane. Draw in the circle, centre C radius CA and read off the x intercepts x_1 and x_2. These are the roots of the quadratic equation.



As an example, for $x^2 - 5x + 4 = 0$, the coordinates of the diameter's endpoints are $A(0,1)$, and $B(5,4)$. The centre C therefore has coordinates $(2.5, 2.5)$. With the compass centred at C open it to a radius of CA and draw in the circle. Read off the circle's x intercepts as $x_1 = 1$ and $x_2 = 4$. You can readily check that these intercepts are the required roots of the quadratic equation.

Why it works

For the given points $A(0,1)$ and $B(p,q)$ we determine $C\left(\frac{p}{2}, \frac{1+q}{2}\right)$ with radius AC given by the equation $r = \frac{1}{2}\sqrt{p^2 + (q-1)^2}$. The circle's equation is then $\left(x - \frac{p}{2}\right)^2 + \left(y - \frac{1+q}{2}\right)^2 = \frac{p^2+(q-1)^2}{4}$.

Setting $y = 0$ and simplifying reveals that the circle intersects the x-axis at $x = \frac{p \pm \sqrt{p^2-4q}}{2}$. These are the roots of the quadratic equation.

If $p = q + 1$ then $x_1 = 1$ and the circle has centre $C\left(\frac{p}{2}, \frac{p}{2}\right)$. This means the circle will be symmetrically positioned across the line $y = x$.

If $p^2 = 4q$ the circle has centre $C\left(\frac{p}{2}, \frac{p^2+4}{8}\right)$ and is tangent to the x axis at $x_1 = \frac{p}{2}$.

Challenge 2: The general Carlyle circle intersects the y axis in at most two places. One of them is $A(0,1)$. Find the other.