The polynomial function $f(x) = 4x^4 - 64x^3 + 359x^2 - 854x + 735$ in its factorised form becomes $f(x) = (x-3)(x-7)(2x-5)(2x-7)$ revealing roots at $x = 3, 7, 2.5$ and 3.5. The average of these roots is 4.

I found using the online program *Wolfram Alpha* that its derivative $f'(x) = 16x^3 - 192x^2 + 718x - 854$ had three roots $x_1 \approx 2.704933..$, $x_2 \approx 3.28111435..$, and $x_3 \approx 6.013952589$ ... and, even though they appeared irrational, I had a healthy suspicion that their average was also 4 also.

The roots of the second derivative $f''(x) = 48x^2 - 384x + 718$ were given as $x_1 = 4 \pm \frac{5}{2\sqrt{6}}$ and $x_2 = 4 \pm \frac{5}{2\sqrt{6}}$, clearly irrational with an average of $\overline{x_i} = 4$. The third derivative, $f'''^{(x)} = 96x - 384$ had the single root of, you guessed it, 4. There was a kind of balance arising with the polynomial function and all of its derivatives, expressed by the one common average.

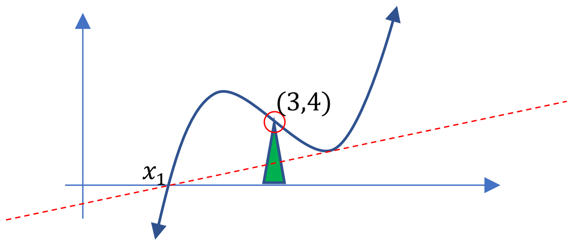
The task of proving this remarkable result for all real valued polynomial functions turned out to be quite straightforward.

For the polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ the sum of the n roots is given by $\sum x_i = \frac{-a_{n-1}}{a_n}$. It follows that the average of those roots is $\overline{x_i} = -\frac{a_{n-1}}{n \cdot a_n}$. Then, because $f'(x) = n \cdot a_n x^{n-1} + a_{n-1}(n-1)x^{n-2} + a_{n-2}(n-2)x^{n-3} + \cdots + a_1$, the corresponding average of the derivative's $(n-1)$ roots is calculated as $\overline{x_i{}'} = -\frac{a_{n-1}(n-1)}{n(n-1) \cdot a_n} = -\frac{a_{n-1}}{n \cdot a_n}$.

It immediately follows that every derivative of the polynomial function must exhibit the same average, since we could call $g(x) = f'(x)$ so that $g'(x) = f''(x)$ etc.

A real valued cubic polynomial with one real root has two complex roots. These complex roots must be complex conjugates because the average of all roots is real. The root of the second derivative of a cubic polynomial locates its inflection point and thus that point also represents the average of the function's roots.

For example, the graph of $y = x^3 - 9x^2 + 25x - 17$ is shown here with a real root at $x_1 = 1$ and complex roots of $x_2 = 4 + i$ and $x_3 = 4 - i$. The inflection is at (3,4).



Challenge 9: A line has been drawn through the real root and tangent to the curve. It can be shown that the x coordinate of the point of contact of the tangent is the real part of the two complex roots. Can you prove it?