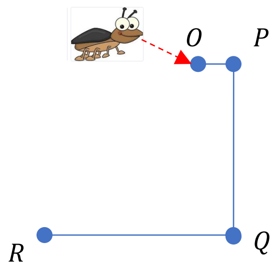
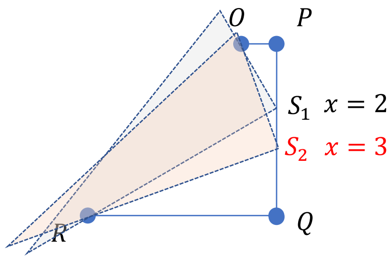
An interesting way to locate the roots of a quadratic equation involves a suitable set square, a sharp pencil and a sheet of paper. The roots are found from a geometric construction and the technique is demonstrated here with the simple quadratic equation $x^2 - 5x + 6 = 0$ (although theoretically real roots can be located using the method for any polynomial of any degree). The coefficients are $1, -5$ and $6$ and these will become the lengths of three connected line segments.

Perhaps the best way to think about Lill's construction is to imagine it as the path made by a smart beetle that can walk forward or backward. We imagine the beetle starting from a point $O$ facing to the right, as shown in the left diagram below. A positive coefficient informs it to move *forward*. Likewise, a negative coefficient informs the beetle to *reverse*. Because, in our example, the coefficient of $x^2$ is 1, the beetle begins by moving one unit forward to $P$. Upon arriving at $P$ the beetle turns 90° anticlockwise as a preparation for its next move. This right-angle rotation ritual is repeated *immediately after* each move.

The coefficient of $x$ is $-5$ and so we now imagine the beetle reversing from $P$ (still facing toward $P$ as it does so) a distance of 5 units downward to a point, say $Q$, whereupon it rotates 90° anticlockwise once again ready for its next move. Finally, the polynomial's constant term 6 sends the beetle forward to the left 6 units to $R$. The beetle again rotates 90° anticlockwise but then stops.




To locate the solutions, slide the set square into a position so that one side of it goes through the point $O$, the right-angled corner of it lies on $PQ$ and the other side goes through $R$. There are in fact only two positions that the set square's corner can be placed, and these correspond to the two roots of the quadratic equation. The first is $S_1$ $(x = 2)$ and by sliding the vertex of the set square downward and readjusting the two edges back onto $O$ and $R$ the second solution is also found at $S_2$ $(x = 3)$Thus, it is the shuffling of the set square that actually solves the quadratic equation!

The two quadratic solutions turn out to be the negative reciprocals of the gradients of $OS_1$ and $OS_2$.

To understand why this is, note that the general equation $ax^2 + bx + c = 0$ can be written as two interlinked linear equations $(ax + b)x + c = Bx + c = 0$ and $B = ax + b$, implying that the solution set $x$ must satisfy both $x = -\frac{c}{B}$ and $x = -\frac{b-B}{a}$. These equal quantities are none other than the gradients of $OS_1$ and $S_1R$ (and similarly for the second root $S_2$) on the two similar triangles $\Delta OPS_1$ and $\Delta S_1QR$ formed by the set square. In our example, $B = -2$ or $-3$.

The method, developed by Austrian engineer Eduard Lill in 1867, can be extended to the general polynomial equation of degree $n$.

*Challenge 4: The diagram for the equation $x^2 + 2x - 3 = 0$ becomes the zig zag line $OPQR$. To solve it, simply extend $PQ$ upward to allow the corner of the set square to lie on it. How are the solutions obtained? For more information click on the website below.*

*https://www.mathematicalwhetstones.com/blog/lills-method-and-horners-scheme*